Sample Solution for HW 2

Constructive Logic

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Problem 1 *Use truth tables to decide whether the following arguments are classically valid:*

(a) Bush will resign or America will go to war if there is a catastrophe.
America is going to war with Bush as president.
There was a catastrophe.
Proof. We can denote the following propositions as follows:

"Bush will resign" — B (we will interpret "Bush is a president" and "Bush will NOT resign" ($\neg B$)
"America will go to war" — A
"There is a catastrophe" — C
Then the argument can be expressed as:

$$\frac{\begin{array}{c} C \to (B \vee A) \\ A \wedge \neg B \end{array}}{C}$$

The the bold row of the following truth table shows that the argument is *invalid.*

A	B	C	$C \to (B \vee A)$	$A \wedge \neg B$	C
T	T	T	T	F	T
T	T	F	T	F	F
T	F	T	T	T	T
T	**F**	**F**	**T**	**T**	**F**
F	T	T	T	F	T
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	T	F	F

■

(b) Bush will resign or America will go to war if there is a catastrophe.
If there is a catastrophe while Bush is president, then America will go to war.
Proof. We can denote the following propositions as follows:

"Bush will resign" — B (we will interpret "Bush is a president" and "Bush will NOT resign" ($\neg B$)

"America will go to war" — A

"There is a catastrophe" — C

Then the argument can be expressed as:

$$\frac{C \to (B \vee A)}{(C \wedge \neg B) \to A}$$

The bold rows of the following truth table shows that the argument is *valid.*

A	B	C	$C \to (B \vee A)$	$(C \wedge \neg B) \to A$
T	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**
T	**F**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**
F	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	*F*	*T*	*F*	*F*
F	**F**	**F**	**T**	**T**

■

Problem 2 *Use Kripke models to prove the following:*

(a) $\Vdash \neg\neg(A \vee \neg A)$

(b) $\neg\neg A \Rightarrow B \Vdash A \Rightarrow \neg\neg B$

Proof. (a) We want to show that for every Kripke model K, $K \Vdash \neg\neg(A \vee \neg A)$. This is an abbreviated way of saying that $\forall i \in K\ i \Vdash \neg\neg(A \vee \neg A)$.

Fix K and let i be any world of K. We want to show that $i \Vdash \neg\neg(A \vee \neg A)$. By the definition of forcing, this is equivalent to saying that $\forall j \geq i,\ j \nVdash \neg(A \vee \neg A)$, which is equivalent to saving that $\exists k \geq j \geq j,\ k \Vdash A \vee \neg A$. In other words, to show that $i \Vdash \neg\neg(A \vee \neg A)$ it suffices to show that for $\forall j \geq j$ we can find a $k \geq j$ such that $k \Vdash A \vee \neg A$. Now let us prove that.

Fix arbitrary $j \geq i$. Now, either there exit a world $l \geq j$ such that $l \Vdash A$ or there is none. (Note that here we use a classical argument — low of excluded middle — to reason about what is true at a world of the model.)

Case 1: There is such $l \Vdash A$. Than, let $k = l$, and we have that $k \Vdash A$, which implies that $k \Vdash A \vee \neg A$ by the definition of forcing.

Case 2: There is no such l. Then $j \Vdash \neg A$, so of $k = j$, $k \Vdash \neg A$, which again implies that $k \Vdash A \vee \neg A$ by the definition of forcing.

Therefore, we can always find $k \geq j$ such that $k \Vdash A \vee \neg A$. This completes the proof. ■

Proof. (b) We want to show that of K is a model such that $K \Vdash \neg\neg A \Rightarrow B$, then also $K \Vdash A \Rightarrow \neg\neg B$.

Fix $K \Vdash \neg\neg A \Rightarrow B$. We want to show that $K \Vdash A \Rightarrow \neg\neg B$, i.e. $\forall i \in K, i \Vdash A \Rightarrow \neg\neg B$. The statement $i \Vdash A \Rightarrow \neg\neg B$ is equivalent to statement $\forall j \geq i,\ j \Vdash A$ implies $j \Vdash \neg\neg B$. As above, the statement $j \Vdash \neg\neg B$ is equivalent to the statement$\forall k \geq j\, \exists l \geq k,\ l \Vdash B$. Therefore, it suffices to show for $j \in K$, if $j \Vdash A$ then $\forall k \geq j\, \exists l \geq k,\ l \Vdash B$.

Fix $j \Vdash A$ (if no such j exist we are trivially done). We claim that $j \Vdash \neg\neg A$. Indeed, by monotonicity $\forall n \geq j, n \Vdash A$, but than $\forall n \geq j \exists m \geq n$ (e.g., n itself) such that $m \Vdash A$. As we sow in the previous proof this is equivalent to $j \Vdash \neg\neg A$. Now we use that fact that $K \Vdash \neg\neg A \Rightarrow B$, and conclude that $j \Vdash B$, by the definition of forcing. By monotonicity, we conclude that $\forall k \geq j\, \exists l \geq k,\ l \Vdash B$ as desired. This concludes that proof. ■

Problem 3 *Show that the following sequent is not derivable in constructive logic:*
$A \Rightarrow B \vdash \neg A \vee B$

Proof. The Soundness Theorem for Kripke semantics states that $A \Rightarrow B \vdash \neg A \vee B$ implies $A \Rightarrow B \Vdash \neg A \vee B$. The counter-positive of this statement is $A \Rightarrow B \nVdash \neg A \vee B$ implies $A \Rightarrow B \nvdash \neg A \vee B$. Thus, to show that the sequent $A \Rightarrow B \vdash \neg A \vee B$ is not derivable it suffices to show that there exit a Kripke model K such that $K \nVdash \neg A \vee B$ but $K \Vdash A \Rightarrow B$. The following model has that property:

$$\begin{array}{ll} \circ^2 & \quad A, B \\ | & \\ \circ_1 & \quad \emptyset \end{array}$$

Clearly,$1 \nVdash \neg A$ and $1 \nVdash B$ so $1 \nVdash \neg A \vee B$. However $1, 2 \Vdash A \Rightarrow B$. ■