# Homework 11

Due December 6 — Practice Final, will not be collected or scored

1. Let $L$ be a language with two unary predicates, $A$ and $B$. Consider the biconditional:

$$\forall x(A(x) \vee B(x)) \leftrightarrow \forall x A(x) \vee \forall x B(x).$$

   (a) Show that one direction is valid, using only semantic notions. In particular, your answer should make it clear that you know what "valid" means!

   (b) Show that the other direction is not valid.

2. Find a prenex formula (i.e. one where all the quantifiers occur up front) equivalent to the following formula:

$$\neg\big(\exists x \forall y R(x,y) \rightarrow \forall z(\exists y A(y) \vee B(z))\big)$$

   Prove the equivalence algebraically.

3. Let $\varphi$ and $\psi$ be any formulas. Give natural deduction proofs of the following formulas (using the 4 quantifier rules, and not defining $\exists\varphi$ as $\neg\forall\neg\varphi$ !).

   (a) $\neg\exists x\varphi(x) \rightarrow \forall x\neg\varphi(x)$

   (b) $\exists x\neg\varphi(x) \rightarrow \neg\forall x\varphi(x)$

   (c) $(\exists x\varphi \rightarrow \psi) \rightarrow \forall x(\varphi \rightarrow \psi)$, where $x$ is not free in $\psi$.

4. Formalize the following argument in first-order logic, and determine whether it is valid (justify your answer).

   Some Greeks are not philosophers.
   No slaves are philosophers.
   Therefore, some Greeks are not slaves.

5. The following problems concern first-order logic. Be sure to answer them in full sentences, defining any symbols used.

   (a) State the Model Existence Lemma.

(b) State the Completeness Theorem.

(c) Assuming the Model Existence Lemma, prove the Completeness Theorem.

6. The language of *linear orders with endpoints* has two constant symbols $0, 1$ and a binary relation symbol, written $x \leq y$. The axioms for linear orders with endpoints are:

   reflexivity: $\forall x\ (x \leq x)$,

   transitivity: $\forall x, y, z\ ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$,

   antisymmetry: $\forall x, y\ ((x \leq y \wedge y \leq x) \rightarrow x = y)$,

   linearity: $\forall x, y\ (x \leq y) \vee (y \leq x)$,

   endpoints: $\forall x\ (0 \leq x) \wedge (x \leq 1)$.

   Consider the following models of the theory of linear orders with endpoints:

   $\mathcal{Q} = ([-1, 1] \subseteq \mathbb{Q}, -1, 1, \leq)$

   (the usual ordering of this rational interval)

   $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, 0, \infty, \leq)$

   (the usual ordering of the natural numbers,
   but with a new element $\infty$ added at infinity)

   (a) Show that these models are distinguishable in first-order logic by producing a sentence that is satisfied by one but not the other.

   (b) A theory $\mathbb{T}$ is said to be *complete* if for every sentence $\alpha$, either $\mathbb{T} \vdash \alpha$ or $\mathbb{T} \vdash \neg\alpha$ and not both. Is the theory of linear orders with endpoints complete?

   (c) Can there be a model that satisfies all the same first-order sentences as $\mathcal{Q}$ and is countable? Justify your answer!

   (d) Are there any models of this theory that are strictly larger than $\mathcal{N}$? Justify your answer!

$\star$ 7. (for Grad Students)

   State and prove the Compactness Theorem for first-order logic. (You may assume other results proved in class.)