Homework 9

Due Thursday, November 15

1. In van Dalen, do problem 3 on p. 85.

2. Let L be any language. Which of the following statements are true and which are false? Justify your answers.

   (a) If φ is any sentence, either $\models \varphi$ or $\models \neg\varphi$.

   (b) If φ is any sentence and $\mathcal{A}$ is any L-structure, either $\mathcal{A} \models \varphi$ or $\mathcal{A} \models \neg\varphi$.

   (c) If φ and ψ are any sentences, $\models \varphi \wedge \psi$ implies $\models \varphi$ and $\models \psi$.

   (d) If φ and ψ are any sentences, $\models \varphi \vee \psi$ implies $\models \varphi$ or $\models \psi$.

3. Find a prenex sentence (i.e. one where all the quantifiers occur up front) logically equivalent to

$$(\exists x A(x) \wedge y = t) \rightarrow \forall y(B(y) \vee \exists x R(y, x)).$$

4. Suppose the formula ψ has no free x. Show that

$$\forall x \varphi(x) \rightarrow \psi \vdash \forall x(\varphi(x) \rightarrow \psi)$$

   does *not* hold in general (give specific formulas $\varphi(x)$ and ψ for which it fails).

5. Show that the following formula cannot be proved:

$$P(a) \vee \ \forall x(R(x, a) \rightarrow \exists y(P(x) \wedge R(x, y)))$$

⋆ 6. Using just the language of equality $=$, give formulas expressing the following conditions on structures (i.e. sets) A:

   (a) A is not empty.

   (b) A has at least n elements (for an arbitrary natural number n).

   (c) A has at most n elements (for an arbitrary natural number n).