Homework 4

Due Thursday, September 27

Recall the following *Well-Definedness Principle* for quotient sets.
Let X be a set with an equivalence relation, written $x \sim y$. The *quotient set*

$$X/\!\sim\ = \{[x] \mid x \in X\}$$

is the set of all *equivalence classes*

$$[x] = \{y \mid x \sim y\}\,.$$

Suppose given a set A, and a function $f : X \to A$ that *respects* $\sim$, in the sense that $f(x) = f(y)$ whenever $x \sim y$. Then there is a (uniquely determined) function on the quotient set,

$$\tilde{f} : X/\!\sim\ \longrightarrow A$$

such that for all $x \in X$,

$$\tilde{f}([x]) = f(x).$$

1. Recall that a (positive) rational number is an equivalence class of pairs of natural numbers $\frac{n}{m} = [(n,m)]$ (where $m \neq 0$), under the equivalence relation
$$(n,m) \sim (n',m') \iff n \cdot m' = n' \cdot m\,.$$

   (a) Show that the function $inv : \mathbb{Q} \to \mathbb{Q}$ giving the multiplicative inverse of a rational number, defined as $inv(\frac{n}{m}) = (\frac{m}{n})$, is well-defined.

   (b) Define the product of a natural number k and a rational number $\frac{n}{m}$ and by:
   $$k \cdot \frac{n}{m} = \frac{k \cdot n}{m}\,.$$
   Is this well-defined?

   (c) Show that the attempted definition of a function $add : \mathbb{Q} \to \mathbb{Q}$ as $add(\frac{n}{m}) = n + m$ is not well-defined.

2. The *Lindenbaum-Tarski algebra* of propositional logic $LT(Prop)$ is by definition the quotient set

$$LT(Prop) = (PROP/\equiv)$$

of the set $PROP$ of propositional formulas, modulo logical equivalence $\varphi \equiv \psi$ (together with a Boolean algebra structure). The elements of $LT(Prop)$ are thus equivalence classes $[\varphi]$, with $[\varphi] = [\psi]$ iff $\varphi \equiv \psi$. The Boolean operations on $LT(Prop)$ are defined by:

$$\begin{aligned} 1 &= [\top] \\ 0 &= [\bot] \\ -[\varphi] &= [\neg\varphi] \\ [\varphi] \sqcap [\psi] &= [\varphi \wedge \psi] \\ [\varphi] \sqcup [\psi] &= [\varphi \vee \psi] \end{aligned}$$

Prove that this *is* a Boolean algebra. (Hint: you may use anything proved in class, but cite the results used. The point is to show that the operations are all well-defined and the equations hold.)

3. Do problems 1d,e,f on page 37 of van Dalen.

⋆ 4. On page 37, do problem 3d.