Homework 1

Due Thursday 6 September

Undergraduates should do only the unstarred problems. Graduate students should also do the starred problem.

1. Prove by induction that if $n \geq 4$, then $n! > 2^n$.
   (Recall that the *factorial function* is defined by $n! = n \cdot (n-1) \cdot \ldots \cdot 1$.)

2. Show that for inductively defined sets, $C_* \subseteq C^*$ where the former is the set of elements having a construction sequence and the latter is the intersection of all inductive sets.

3. Define the set of "babble-strings" inductively, as follows:

   - the expression "ba" is a babble-string
   - if s is a babble-string, so is "ab"$\hat{\ } s$
   - if s and t are babble-strings, so is $s\hat{\ }t$

   Here $\hat{\ }$ is the *concatination operation*: $x\hat{\ }y = xy$.

   Prove by induction that every babble-string has the same number of a's and b's, and that every babble-string ends with an "a".

4. Referring to the previous problem, show that the set B of babble-strings is not freely generated. Give a different specification of (the same set) B such that B is freely generated. Use that specification to define a length function $f : B \to \mathbb{N}$ giving the number of letters in the string.

⋆ 5. The set of (unlabelled, binary) trees is defined inductively as follows:

   - $*$ is a tree
   - if s, t are trees, so is $[s, t]$

   Define functions on trees that count the height of a tree and the width of a tree (the latter is the same as the number of branch-ends). Prove that the height of a tree is always less than or equal to the width of the tree.