Solutions to Homework #9

2. a. Free; $x = x$
   b. Free; $y = y$
   c. Free; $z = \bar{0}$
   d. Free; $\exists x\ (\bar{0} + y = x)$
   e. Free; $\exists w\ (w + x = \bar{0})$
   f. Not free; $\forall w\ (x + (x + w) = \bar{0})$
   g. Not free; $\forall w\ (x + (x + y) = \bar{0}) \wedge \exists y\ (x + y = x)$
   h. Free; $\forall u\ (u = v) \rightarrow \forall z\ (z = y)$

3. a. There are lots of terms that denote 5 in this structure; for example, $S(S(S(S(S(c)))))$ and $(S(S(S(S(S(c))))) + c)$.
   b. To denote n, just use the term $S(S(S(\ldots S(c))))$, with n S's in all. More formally, use induction on n; if $n = 0$, use c; and if t denotes n, $S(t)$ denotes $n + 1$.
   c. If t denotes n, then so do $(t + c)$, $((t + c) + c)$, $(((t + c) + c) + c)$, and so on.

6. a. To show $\exists x\ (\varphi(x) \wedge \psi(x)) \rightarrow (\exists x\ \varphi(x) \wedge \exists x\ \psi(x))$ is valid, I need to show that it is true in every structure $\mathfrak{A}$. I will use Lemma 2.4.5 repeatedly.

   Saying that the formula above is true in $\mathfrak{A}$ amounts to saying that if $\exists x\ (\varphi(x) \wedge \psi(x))$ is true in $\mathfrak{A}$ then $(\exists x\ \varphi(x) \wedge \exists x\ \psi(x))$ is true in $\mathfrak{A}$. To show this, suppose $\exists x\ (\varphi(x) \wedge \psi(x))$ is true in $\mathfrak{A}$. By Lemma 2.4.5, this means that there is some a in the universe of $\mathfrak{A}$ such that $\varphi(\bar{a}) \wedge \psi(\bar{a})$ is true in $\mathfrak{A}$. For this value of a, both $\varphi(\bar{a})$ and $\psi(\bar{a})$ are true in $\mathfrak{A}$. But this means that $\exists x\ \varphi(x)$ is true in $\mathfrak{A}$, and $\exists x\ \psi(x)$ is true in $\mathfrak{A}$, and hence $\exists x\ \varphi(x) \wedge \exists x\ \psi(x)$ is true in $\mathfrak{A}$, which is what I needed to show.

   Note that saying "θ is true in $\mathfrak{A}$" is synonymous with "$\mathfrak{A} \models \theta$"; which phrasing you use is a matter of taste.

   b. To show that the other direction is not valid, it's enough to describe a counterexample. Let L be a language with two constant symbols a and b, and let $\mathfrak{A}$ be a structure in which a and b denote different objects (say, the structure with universe $\{0, 1\}$, where $a^{\mathfrak{A}} = 0$, $b^{\mathfrak{A}} = 1$). Then $\mathfrak{A} \models \exists x\ (x = a) \wedge \exists x\ (x = b)$, but $\mathfrak{A} \not\models \exists x\ (x = a \wedge x = b)$. (There are many other simple counterexamples.)

7. a. This is false. For example, let φ be the formula $\forall x \ \forall y \ R(x,y)$. Then φ is true in a structure $\mathfrak{A}$ where the denotation of R holds of everything (for example, take $\langle \mathbb{N}, Q \rangle$ where Q holds of every pair of natural numbers), and φ is false in a structure $\mathfrak{B}$ where there is at least one pair of values for which the denotation of R does not hold (for example, take $\langle \mathbb{N}, <^{\mathbb{N}} \rangle$).

   b. This is true, by Lemma 2.4.5: for any structure $\mathfrak{A}$, $\mathfrak{A} \models \neg\varphi$ iff $\mathfrak{A} \not\models \varphi$.

   c. This is false. For example, let Γ be the empty set, and use part (a).