Solutions to Homework #12

2.

$$
\dfrac{[\exists x\ \theta(x)]_2 \qquad \dfrac{[\theta(x)]_1 \qquad \dfrac{\forall x\ (\theta(x) \to \eta)}{\theta(x) \to \eta}}{\eta}}{\dfrac{\eta}{\exists x\ \theta(x) \to \eta}\ 2}\ 1
$$

3.

a.

$$
\dfrac{\dfrac{\dfrac{[\exists x\ (\varphi \to \psi(x))]_3 \qquad \dfrac{\dfrac{[\varphi \to \psi(x)]_1 \qquad [\varphi]_2}{\psi(x)}}{\exists x\ \psi(x)}}{\exists x\ \psi(x)}\ 1}{\varphi \to \exists x\ \psi(x)}\ 2}{(\exists x\ (\varphi \to \psi(x))) \to (\varphi \to \exists x\ \psi(x))}\ 3
$$

b.

$$
\dfrac{\exists x\ \psi(x) \qquad \dfrac{\dfrac{[\psi(x)]_1}{\varphi \to \psi(x)}}{\exists x\ (\varphi \to \psi(x))}}{\exists x\ (\varphi \to \psi(x))}\ 1
$$

c.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg \exists x\ \psi(x) \qquad \dfrac{\varphi \to \exists x\ \psi(x) \qquad [\varphi]_1}{\exists x\ \psi(x)}}{\bot}}{\psi(x)}}{\varphi \to \psi(x)}\ 1}{\exists x\ (\varphi \to \psi(x))}
$$

d. This is just the $\vee$ elimination rule:

$$
\dfrac{\dfrac{\exists x\ \psi(x) \vee \neg \exists x\ \psi(x) \qquad \begin{array}{c} [\exists x\ \psi(x)]_1 \\ \vdots (b) \\ [\exists x\ (\varphi \to \psi(x))]_2 \end{array} \qquad \begin{array}{c} [\neg \exists x\ \psi(x)]_1 \qquad \varphi \to \exists x\ \psi(x) \\ \vdots \qquad\qquad \vdots (c) \\ \exists x\ (\varphi \to \psi(x)) \end{array}}{\exists x\ (\varphi \to \psi(x))}\ 1}{(\exists x\ (\varphi \to \psi(x)) \to \exists x\ (\varphi \to \psi(x)))}\ 2
$$

5. Suppose T is a maximally consistent theory.

   For the forwards direction, suppose φ is in T. Since T is consistent, $\neg\varphi$ is not in T.

   For the other direction, suppose $\neg\varphi$ is not in T. By maxmimality, $T \cup \{\neg\varphi\}$ is inconsistent. So there is a proof of $\perp$ from T and $\neg\varphi$. Using RAA, we get a proof of φ from T. Since T is a theory, φ is in T.

11. Suppose T_1 is a conservative extension of T_2, and T_2 is a conservative extension of T_3. I need to show that T_1 is a conservative extension of T_3. In other words, I need to show that if φ is any sentence in L_3, then φ is in T_1 if and only if it is in T_3.

    Since T_1 contains T_2 and T_2 contains T_3, it is clear that every sentence φ in T_3 is in T_1. For the other direction, suppose φ is some sentence in the language L_3 that is in T_1. Since L_3 is a smaller language than L_2, φ is also a sentence in L_2. Since T_1 is a conservative extension of T_2, φ is in T_2. And since T_2 is a conservative extension of T_3, then φ is in T_3, as required.

13. a. Let $f(x) = 1 - x$. This is an isomorphism of the two structures, as follows. *f is injective:* if $1 - x = 1 - y$ then $x = y$. *f is surjective:* Given any z in $(0, 1)$, $z = f(1 - z)$. *f is an isomorphism:* If $a < b$ then $1 - a > 1 - b$.

    b. Let $f(x) = x/(1 - x)$. *f is injective:* if $x/(1 - x) = y/(1 - y)$, then (cross multiplying) we have $x - xy = y - xy$ and so $x = y$. *f is surjective:* if z is any positive real number, let $x = z/(1 + z)$. Then x is an element of $(0, 1)$, and it is easy to check that $f(x) = z$. *f is an isomorphism:* Assuming x and y are in $(0, 1)$, $1 - x$ and $1 - y$ are both positive. So we have $x/(1 - x) < y/(1 - y)$ iff $x - xy < y - xy$ iff $x < y$.

    c. $[0, 1]$ satisfies "there is a smallest element," $\exists x \; \forall y \; (x \leq y)$, while $(0, 1)$ does not.

14. a. The structure $\mathcal{B}$ in the problem mentioned ordered the natural numbers so that all the even numbers come first, followed by the odd numbers:

    $$0, 2, 4, 6, \ldots, 1, 3, 5, 7, 9 \ldots$$

    Let X be any nonempty subset of the universe of $\mathcal{B}$. If X has any even numbers, take the smallest even number in X, under the usual ordering on $\mathbb{N}$; this is the least element of X in the ordering on $\mathcal{B}$. Otherwise, if there are no even numbers in X, there is at least one odd number in X. In that case, the smallest odd number in X, under the usual ordering on $\mathbb{N}$, is the least element of X in the ordering on $\mathcal{B}$.

b. Let Γ be a set of sentences, such that every well-ordering is a model of Γ. Using compactness, I will show that there is a structure that is *not* a well-ordering, but is also a model of Γ.

Add constants $c_0, c_1, c_2, \ldots$ to the language. Let Γ' be the set of sentences

$$\Gamma \cup \{c_1 < c_0, c_2 < c_1, c_3 < c_2, \ldots\}.$$

I claim that every finite subset of Γ' is consistent. Let Δ be any such finite subset, and notice that for some n, Δ is a subset of

$$\Gamma \cup \{c_1 < c_0, c_2 < c_1, c_3 < c_2, \ldots, c_n < c_{n-1}\}.$$

In other words, only finitely many of the sentences $c_{i+1} < c_i$ can be in Δ. Since $\langle \mathbb{N}, < \rangle$ is a model of Γ, the structure

$$\langle \mathbb{N}, <, n, n-1, n-2, \ldots, 3, 2, 1, 0, 0, 0, 0 \ldots \rangle$$

is a model of Δ (that is, the structure that assigns n to c_0, $n-1$ to c_1, and so on). Note that the constants from c_{n+1} don't appear in Δ, so we can just assign 0 to them.

Since every finite subset of Γ' has a model, Γ' also has a model $\mathcal{A}'$ (in the language with the new constants). Let $\mathcal{A}$ be the reduct of Γ' to the original language. Then $\mathcal{A}$ is a model of Γ, but $\mathcal{A}$ has elements $a_0, a_1, a_2, \ldots$ such that $a_1 < a_0$, $a_2 < a_1$, and so on. Then the set

$$\{a_0, a_1, a_2, \ldots, \}$$

doesn't have a least element, so $\mathcal{A}$ is not a well-ordering.