Homework #13

Due Wednesday, December 5

1. Read Chapter 4 of van Dalen, and as much of Chapter 5 as you can.

2. Note that this is the last homework assignment to be turned in! The final exam is on Friday, December 14, from 1-3 PM. Next week, I will announce extra office hours. On the exam, I will only test you directly on material covered since the midterm. But note that this includes, implicitly, a lot of material from the first half of the course, such as the notion of a maximally consistent set, proof rules for the propositional connectives, etc.

3. Do problems 7–10 on page 119.

⋆ 4. Do problem 13 on page 119. (Note that saying that $Mod(T_1 \cup T_2) = \emptyset$ is equivalent to saying that $T_1 \cup T_2$ is inconsistent. Use the compactness theorem.)

⋆ 5. Show that if T_1 and T_2 are theories, and $T_1 \neq T_2$, then $Mod(T_1) \neq Mod(T_2)$. In other words, if $T_1 \neq T_2$, then there is a structure that is a model of one but not the other. (Hint: show that if $T_1 \neq T_2$, there is a sentence φ in one but not the other. Without loss of generality, say φ is in T_1 but not T_2. Using the fact that T_2 is a theory, show $T_2 \cup \{\neg\varphi\}$ is consistent.)

∘ 6. Do problem 3 on page 133. This is a nice application of compactness.

7. Do problem 5 on page 134.

⋆ 8. Do problem 6 on page 134. Note that $\mathcal{A} \subseteq \mathcal{B}$ means that $\mathcal{A}$ is a substructure of $\mathcal{B}$, and $\mathcal{A} \prec \mathcal{B}$ means that $\mathcal{A}$ is an *elementary* substructure of $\mathcal{B}$.

9. What subsets of the real numbers are first-order definable in the structure $\langle \mathbb{R}, < \rangle$?

⋆ 10. Show that multiplication (that is, the relation $x \times y = z$) is not definable in $\langle \mathbb{R}, 0, +, < \rangle$. (Hint: find an automorphism f of this structure, such that for some a and b $f(a \times b)$ is not equal to $f(a) \times f(b)$.)

11. Show that addition is not definable in the structure $\langle \mathbb{N}, \times \rangle$. (Hint: consider an automorphism that switches two primes.)

12. Explain Skolem's paradox, and why it isn't really a paradox.

$\star$ 13. Let T be a complete theory with an effective set of axioms (in other words, there is an algorithm which determines if a given string of symbols is an axiom of T). Show that T is decidable (that is, there is an algorithm which determines whether or not a given string of symbols is in T, i.e. provable from the axioms).

14. The "theory of a successor operation" is the theory in the language $0, S$ axiomatized by the following sentences:

    - $\forall x\ (\neg S(x) = 0)$
    - $\forall x, y\ (S(x) = S(y) \rightarrow x = y)$
    - For each i, the sentence $\forall x\ \neg S^i(x) = x$

    The last item is a schema; the notation $S^i(x)$ means $S(S(\ldots S(x)))$ where S occurs i times.

    a. What does a model of this theory look like?

    b. Show that this theory is not categorical for countable structures.

    $\circ$ c. Show that this theory *is* categorical for uncountable structures, and hence, by the Los-Vaught test, complete.

$\star$ 15. Let L be the language with a single binary relation $<$. Show the the class of well-orderings is definable in second-order logic.

$\circ$ 16.

    a. Let L be the language with no "built-in" function and relation symbols other than equality. Find a formula φ in the language of second-order logic, such that for every (full) structure $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ if and only if $|\mathfrak{A}|$ is infinite. In other words, show that the class of infinite structures is definable by a single formula in second-order logic. (Hint: use the suggestions in the notes to express the assertion that there is an injective map from the universe to a proper subset of itself.)

    b. Show that the class of finite structures is definable in second-order logic.

    c. Show that compactness does not hold for second-order logic, by exhibiting a set of sentences which is finitely satisfiable, but not satisfiable.