Homework #12

Due Wednesday, November 28

1. Continue reading sections 3.2 and 3.3 in van Dalen, in conjunction with the class notes.

⋆ 2. Assuming $\theta(x)$ and η are any formulas and x is not free in η, prove $\exists x\ \theta(x) \to \eta$ from $\forall x\ (\theta(x) \to \eta)$.

⋆ 3. Suppose φ and $\psi(x)$ are any formulas, and x is not free in φ. Prove

$$(\varphi \to \exists x\ \psi(x)) \leftrightarrow \exists x\ (\varphi \to \psi(x))$$

using the following steps:

a. First prove the $\leftarrow$ direction.

b. From $\exists x\ \psi(x)$, prove $\exists x\ (\varphi \to \psi(x))$.

c. From $\neg\exists x\ \psi(x)$ and $\varphi \to \exists x\ \psi(x)$, conclude $\exists x\ (\varphi \to \psi(x))$. (Hint: from the hypotheses, show that φ implies *anything*.)

d. Put parts (b) and (c) together with a proof of $\exists x\ \psi(x) \vee \neg\exists x\ \psi(x)$ (you don't have to write out the latter) to obtain a proof the $\to$ direction.

Note that this $\to$ direction of this problem, together with problem 11, are used in the proof of van Dalen's Lemma 3.1.7.

4. Show that any maximally consistent set of sentences is a theory.

⋆ 5. Suppose T is a maximally consistent theory. Prove that φ is in T if and only if $\neg\varphi$ is not in T.

6. Suppose T is a maximally consistent theory. Prove that $\varphi \to \psi$ is in T if and only if either φ is not in T, or ψ is in T. You may use the previous two problems.

∘ 7. Do problem 1 on page 112.

8. Do problem 2 on page 112.

∘ 9. Do problems 3 and 4 on page 112.

10. Consider the following two statements of the completeness theorem:

Version A: If Γ is any consistent set of sentences, then Γ has a model.

Version B: If Γ is any set of sentences, φ is any sentence, and $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

Show *directly* that these two statements are equivalent, i.e. that each one implies the other. (Hint: to show that B implies A, take φ to be $\perp$.)

$\star$ 11. Let L_1, L_2, and L_3 be languages, with $L_1 \supseteq L_2 \supseteq L_3$. (In other words, L_1 has all the constant, function, and relation symbols of L_2, and possibly more; and similarly for L_2 and L_3.) Suppose T_1, T_2, and T_3 are theories in the languages L_1, L_2, and L_3 respectively. Show that if T_1 is a conservative extension of T_2, and T_2 is a conservative extension of T_3, then T_1 is a conservative extension of T_3. (You can find the definition of "conservative extension" on page 106 in van Dalen, Definition 3.1.5.)

12. Do problems 1 through 5 on page 118–119.

$\star$ 13. Let $(0,1)$ denote the open interval of real numbers between 0 and 1:

$$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}.$$

Let $[0,1]$ denote the closed interval

$$[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}.$$

Let $(0,\infty)$ denote the positive real numbers,

$$(0,\infty) = \{x \in \mathbb{R} \mid x > 0\}.$$

a. Show that $\langle (0,1), < \rangle$ is isomorphic to $\langle (0,1), > \rangle$, by exhibiting a bijective function from $(0,1)$ to $(0,1)$ and proving that it is an isomorphism of the two structures. Note that the underlying language has a single binary relation r that is interpreted as $<$ in the first structure and $>$ in the second.

b. Show that $\langle (0,1), < \rangle$ is isomorphic to $\langle (0,\infty), < \rangle$. (Hint: consider the function $f(x) = \frac{x}{1-x}$.)

c. Show that $\langle (0,1), < \rangle$ is *not* isomorphic to $\langle [0,1], < \rangle$. (Hint: use Lemma 3.3.3 in van Dalen, and find a sentence that is true in one structure but false in the other.)

$\star$ 14. Let $\mathcal{P} = \langle P, < \rangle$ be a linear ordering. $\mathcal{P}$ is said to be a *well-ordering* if every nonempty subset of P has a least (minimum) element. Note that $\langle \mathbb{N}, < \rangle$ has this property, so you can think of elements of a well-ordering as "generalized numbers" (a.k.a. "ordinals").

a. Show that the structure $\mathcal{B}$ in exercise 14 on page 91 of van Dalen is a well-ordering.

b. Do problem 6 on page 119. In other words, use the suggestion to show that there is no set of sentences Γ such that the models of Γ are exactly the well-orderings.