Econ 73-250A-F
Spring 2001
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Suggested Answers to Problem Set #4

Please refer to the end of the document for all diagrams.

Exercise 18.0. First part:

$f(x_1,x_2)$	$MP_1(x_1,x_2)$	$MP_2(x_1,x_2)$	$TRS(x_1,x_2)$
x_1+2x_2	1	2	$-\frac{1}{2}$
ax_1+bx_2	a	b	$-\frac{a}{b}$
$50x_1x_2$	$50x_2$	$50x_1$	$-\frac{x_2}{x_1}$
$x_1^{1/4}x_2^{3/4}$	$\frac{1}{4}x_1^{-3/4}x_2^{3/4}$	$\frac{3}{4}x_1^{1/4}x_2^{-1/4}$	$-\frac{x_2}{3x_1}$
$Cx_1^ax_2^b$	$Cax_1^{a-1}x_2^b$	$Cbx_1^ax_2^{b-1}$	$-\frac{ax_2}{bx_1}$
$(x_1+2)(x_2+1)$	x_2+1	x_1+2	$-\frac{x_2+1}{x_1+2}$
$(x_1+a)(x_2+b)$	x_2+b	x_1+a	$-\frac{x_2+b}{x_1+a}$
$ax_1+b\sqrt{x_2}$	a	$\frac{b}{2}x_2^{-\frac{1}{2}}$	$-\frac{2a}{bx_2^{-\frac{1}{2}}}$
$x_1^a+x_2^a$	ax_1^{a-1}	ax_2^{a-1}	$-\frac{x_1^{a-1}}{x_2^{a-1}}$
$(x_1^a+x_2^a)^b$	$bax_1^{a-1}(x_1^a+x_2^a)^{b-1}$	$bax_2^{a-1}(x_1^a+x_2^a)^{b-1}$	$-\frac{x_1^{a-1}}{x_2^{a-1}}$

Second part:

$f(x_1,x_2)$	Scale	MP_1	MP_2
x_1+2x_2	C	C	C
$\sqrt{x_1+2x_2}$	D	D	D
$.2x_1x_2^2$	I	C	I
$x_1^{1/4}x_2^{3/4}$	C	D	D
$x_1+\sqrt{x_2}$	D	C	D
$(x_1+1)^{.5}(x_2)^{.5}$	D	D	D
$\left(x_1^{\frac{1}{3}}+x_2^{\frac{1}{3}}\right)^3$	C	D	D

Exercise 18.4. The production function is

$$f(K,L)=\frac{L}{2}+\sqrt{K}.$$

(a) There are decreasing returns to scale because, for $t>1$

$$f(tK,tL)=L+\sqrt{t}\sqrt{K}<L+t\sqrt{K}=tf(K,L).$$

The marginal product of labor is constant and equal to 0.5.

(b) In the short run $K = 4$. Output as function of L is

$$y = f(4,L) = \frac{L}{2} + \sqrt{4} = \frac{L}{2} + 2.$$

The average product of labor is

$$\frac{f(K,L)}{L} = \frac{1}{2} + \frac{2}{L}.$$

See Figure 1.

Exercise 19.2. (a) The firm's profits as function of the amount of input x are

$$\Pi = (100)\,4\sqrt{x} - 50x$$

(b) The profit-maximizing amount of input is obtained by setting

$$\frac{\partial \Pi}{\partial x} = \frac{1}{2}(100)\,4x^{-\frac{1}{2}} - 50 = 0.$$

Solve this with respect to x to get

$$x^* = 16.$$

Output is

$$y^* = 4\sqrt{x^*} = 16.$$

Profits are

$$\Pi^* = (100)\,16 - (50)\,16 = (50)\,16 = 800.$$

(c) The firm maximizes

$$\begin{aligned} \Pi &= (100)\,4\sqrt{x} - 50x - (20)\,4\sqrt{x} + 10x \\ &= (80)\,4\sqrt{x} - 40x. \end{aligned}$$

The first order condition is

$$\frac{\partial \Pi}{\partial x} = \frac{1}{2}(80)\,4x^{-\frac{1}{2}} - 40 = 0.$$

Solve for the optimal amount of input

$$x^* = 16.$$

Output is

$$y^* = 4\sqrt{x^*} = 16.$$

Profits are

$$\Pi^* = (80)\,4\sqrt{16} - (40)\,16 = (40)\,16 = 640.$$

(d) The after tax profits as a function of the amount of input are

$$\Pi = 0.5\left((100)\,4\sqrt{x} - 50x\right).$$

The input choice, as well as output, are the same as in (b). Profits are 50% of what they were in (b), or 400.

Exercise 19.9. (a) Farmer Hoglund will choose fertilizer in an amount at which the marginal product of fertilizer multiplied by the price of corn is equal to its cost

$$3\left(1-\frac{N}{200}\right)=p.$$

This condition is the first order condition for profit maximization. Solving for the optimal amount we get

$$N_H^*=200\left(1-\frac{p}{3}\right).$$

(c) Let the production function of Hoglund be

$$y=f_H(N).$$

Then the production function of Skoglund, $f_S(N)$, is

$$f_S(N)=2f_H(N).$$

Therefore, the marginal product of fertilizer for Skoglund is

$$\frac{\partial f_S(N)}{\partial N}=\frac{\partial(2f_H(N))}{\partial N}=2\frac{\partial f_H(N)}{\partial N},$$

where we know that

$$\frac{\partial f_H(N)}{\partial N}=\left(1-\frac{N}{200}\right).$$

Then,

$$\frac{\partial f_S(N)}{\partial N}=2\left(1-\frac{N}{200}\right).$$

The optimal choice of fertilizer for Skoglund is found by setting the marginal product of fertilizer times the corn price equal to its cost

$$(3)\,2\left(1-\frac{N}{200}\right)=p.$$

Solve for N_S^*

$$N_S^*=200\left(1-\frac{p}{6}\right).$$

(d) We want to know whether

$$\begin{aligned}
f_S(N_S^*) &> 2f_H(N_H^*) \text{ or}\\
f_S(N_S^*) &= 2f_H(N_H^*) \text{ or}\\
f_S(N_S^*) &< 2f_H(N_H^*).
\end{aligned}$$

We know that

$$f_S(N_S^*) = 2f_H(N_S^*).$$

We also know, from points (c) and (b), that

$$N_S^* > N_H^*.$$

Therefore Skoglund produces more than twice as much output as Hoglund, because:

$$f_S(N_S^*) = 2f_H(N_S^*) > 2f_H(N_H^*).$$

(e) This question is not very clear. First of all it is not clear whether the productivity of fertilizer is higher for Skoglund or Hoglund. That is, for p close enough to zero it is the case that $N_S^* = N_H^* = 200$ so that

$$\frac{f_S(N_S^*)}{N_S^*} > \frac{f_H(N_H^*)}{N_H^*},$$

while for p close enough to 3, $N_S^* > N_H^* = 0$, so it is the case that

$$\frac{f_S(N_S^*)}{N_S^*} < \frac{f_H(N_H^*)}{N_H^*} = +\infty.$$

Second, and most important, even if we knew that

$$\frac{f_S(N_S^*)}{N_S^*} > \frac{f_H(N_H^*)}{N_H^*},$$

as the exercise seems to suggest, we could not conclude much about the productivity of fertilizer.

A person that could not observe the quality of their land would think that the farmers are using the same technology $f(N)$. A person that could observe the yields and the use of fertilizer, might conclude that the productivity of fertilizer is higher the higher the utilization of fertilizer. This must imply that the marginal product of fertilizer for both farmers *increases* with N:

$$\frac{\partial f(N)}{\partial N} \text{ increases with } N.$$

There are two problems with this type of reasoning: 1) if the farmers have the same technology, what explains the fact that they use different amounts of inputs? It must then be that one of them is not maximizing profits. 2) If the marginal product of fertilizer increases with N, then the optimal choice of fertilizer is infinite! To see this, consider the point at which

$$3\frac{\partial f(N^*)}{\partial N} = p.$$

At this point, a farmer would be minimizing rather than maximizing profits. In fact, by using an extra unit of N the farmer can increase its marginal product. Thus, for every $N > N^*$:

$$3\frac{\partial f(N)}{\partial N} > p,$$

the marginal contribution to the farmer's revenue of a marginal unit of fertilizer exceeds the fertilizer's price. Thus, a profit maximizing farmer should buy an infinite amount of fertilizer.

FIGURE 1