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# Weights

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**Abstract.** Algebras of shapes, $U_{ij}$, can be augmented with labels and weights to form new algebras, $V_{ij}$ and $W_{ij}$, in which computations are defined by shape grammars.

**Shapes**

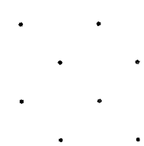
The algebras in the table

$$\begin{array}{cccc} U_{00} & U_{01} & U_{02} & U_{03} \\ & U_{11} & U_{12} & U_{13} \\ & & U_{22} & U_{23} \\ & & & U_{33} \end{array}$$

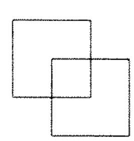
provide the main objects and devices used in shape grammars. An algebra $U_{ij}$ contains shapes. Every shape is a finite but possibly empty set of basic elements that are maximal with respect to one another.

Basic elements are points, lines, planes, or solids that are defined in dimension $i = 0, 1, 2,$ or 3, and combined and manipulated in dimension $j \geqslant i$. For $i > 0$, they have finite, nonzero content—either length, area, or volume—and boundaries that are shapes in the algebra $U_{i-1,i}$. Basic elements are connected: ones of dimension $i > 0$ cannot be divided by others of dimension less than $i-1$. Points divide lines, lines divide planes, and planes divide solids.

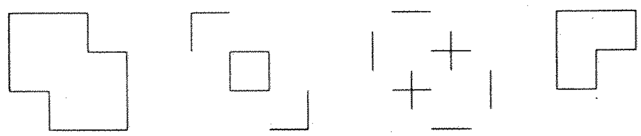
Basic elements are arranged to make shapes, and are assumed to be maximal in combination. This allows for shapes to be identified uniquely by sets of basic elements with definite members that are independent of one another, and, at the same time, to be without definite parts in algebras where $i > 0$. Consider shapes in the algebras $U_{02}$, $U_{12}$, and $U_{22}$ that are defined in the plane with points, lines, and planes. A set containing eight points forms the shape



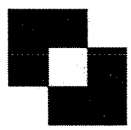
in the algebra $U_{02}$, and, because points are not divisible, the subsets of this set exhaust all of the parts of the shape. Shapes in the other algebras, however, are more complicated. A set containing eight lines forms the shape



in the algebra $U_{12}$, but now the subsets of this set do not exhaust all of the parts of the shape. The shape has infinitely many other parts like the shapes shown here



that contain segments of the lines. And a set of two planes forms the shape



in the algebra $U_{22}$ with infinitely many parts, including these

The shapes in an algebra $U_{ij}$ are partially ordered by a part relation $\leqslant_U$, and combined by Boolean operations of sum $+_U$ and difference $-_U$. This forms a relatively complemented, distributive lattice. The shape given by the empty set is the zero, but there is no unit except in $U_{00}$. (Equivalently, a Boolean ring is formed with the operations of symmetric difference and product that are defined in terms of $+_U$ and $-_U$.) The algebra $U_{ij}$ is also closed under the Euclidean transformations of dimension $j$.

The exact details of maximality, the part relation, and the operations of sum and difference can be developed in alternative ways. In one approach, embedding serves as a primitive relation on basic elements, where, for example, one line is embedded in another line if the one is a segment of the other. Embedding is used both to decide when basic elements are maximal in combination, and to define the part relation on shapes. The part relation in turn leads directly to definitions of sum and difference. This approach is neat, but it is not constructive. Another approach allows for the explicit definition of sets of maximal elements for arbitrary sets of basic elements, and for sums and differences. This is done with reduction rules that are framed in terms of the embedding relation on basic elements. Either sum or difference then provides another definition of the part relation.

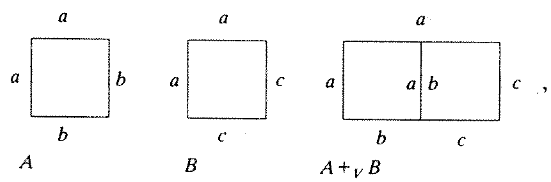
**Shapes and labels**

More complex algebras containing shapes made up of basic elements and other things are also possible. An algebra $U_{ij}$ can be augmented by associating labels from a given vocabulary with basic elements to classify them in shapes, or to introduce additional information. This allows for shapes to be formed as collections of others that are distinguished by distinct labels. Members of different collections interact if they are distinguished by the same label, and are independent otherwise. A new algebra $V_{ij}$ is so defined that preserves the Boolean properties of $U_{ij}$. And, $V_{ij}$ has the Euclidean properties of $U_{ij}$, too, if labels are invariant under the transformations.

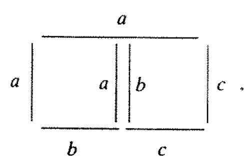
The description of an algebra $V_{ij}$ is familiar when $i = 0$. In this case, $V_{0j}$ contains sets of labeled points. More generally, $V_{ij}$ contains shapes, each being a finite but possibly empty set of ordered pairs $(e, a)$. These contain a basic element $e$ of the algebra $U_{ij}$, and a label $a$, so that basic elements with identical labels are maximal.

The shapes in an algebra $V_{ij}$ are partially ordered by a part relation $\leqslant_V$, and combined by Boolean operations of sum $+_V$ and difference $-_V$. The needed definitions are framed easily in terms of $\leqslant_U$, and $+_U$ and $-_U$. Let the shape $A_a$ in $U_{ij}$ contain the maximal elements distinguished by the label $a$ in the ordered pairs in a shape $A$ in $V_{ij}$. Conversely, let the shape $A^a$ in $V_{ij}$ contain the ordered pairs obtained when the label $a$ is associated with the maximal elements in a shape $A$ in $U_{ij}$. Then, the part relation $A \leqslant_V B$ holds whenever $A_a \leqslant_U B_a$, for every label $a$ in a vocabulary $L$. Further, the sum $A +_V B$ is given by $\bigcup_{a \in L} (A_a +_U B_a)^a$, and the difference $A -_V B$ is given in the same fashion by $\bigcup_{a \in L} (A_a -_U B_a)^a$.

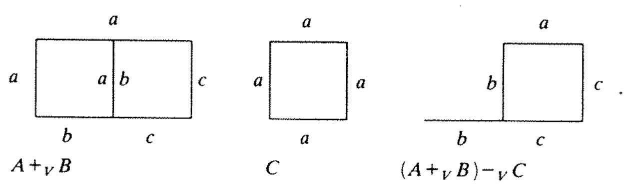
For example, two labeled squares in $V_{12}$ form a sum in this way




that contains the seven labeled lines



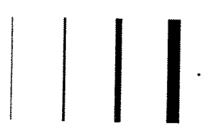

And this sum and another labeled square combine in a difference in this way




**Weights**

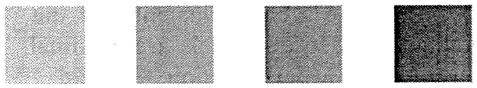
In a more ambitious fashion, properties can be included in shapes, so that basic elements with different properties can interact as desired. The elaboration of this idea in an algebra $U_{ij}$ is illustrated nicely with numerical weights.

Weights are familiar in architecture, graphics, and the visual arts, where lines of different thicknesses are used in drawing

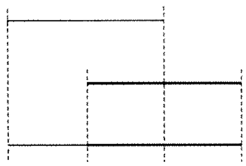


It is easy to extend this device in a variety of ways: to points, for example, when each has an associated radius

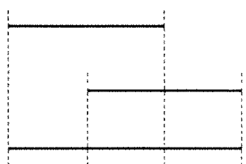
and to planes and solids when each has an interior filled by a uniform tone



The properties of weights determine a simple algebra with a relation and operations corresponding to $\leqslant_U$, and $+_U$ and $-_U$ in an algebra of shapes. Some of these properties are apparent in drawing. Suppose two lines of different thicknesses are drawn at the same time, either one overlapping the other, so that they have an embedded line in common that exhausts neither of them,




The thicker line appears at full length, but the thinner one is shortened. And if the two lines have equal thickness, then a single line is formed,




In both of these cases, the thickness of the embedded line in which two weights combine depends on the thicknesses of the lines, being the greater thickness of the two lines.

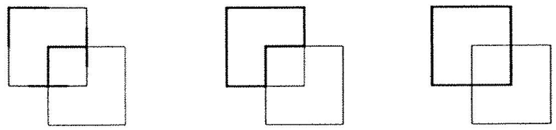
This observation gives a rule to sum weights $u$ and $v$: $u+v = \max\{u, v\}$. The rule has the nice property that the relation $u \leqslant v$ defined by the identity $u+v = v$ determines a partial (linear) order on weights in the usual way.

How weights combine in difference is not as clear as it is for sum, and there is little in drawing to provide guidance. Algebraic considerations, however, suggest an interesting rule. Weights and the relation $\leqslant$ form a lattice in which any two weights $u$ and $v$ have the sum $u+v$ as their least upper bound, and the product $u \cdot v$ given by $\min\{u, v\}$ as their greatest lower bound. If, in Boolean fashion, $u \cdot v = u-(u-v)$, then the rule whereby $u-v$ is the arithmetic difference of $u$ and $v$ if $v \leqslant u$, and zero otherwise is a good choice.

**Shapes and weights**

Weights can be introduced in an algebra $U_{ij}$ to obtain a new algebra $W_{ij}$. These algebras are structured in different ways. However, the lattice properties of $U_{ij}$, including distributivity, are preserved in $W_{ij}$. And, $U_{ij}$ is isomorphic to certain subsets of $W_{ij}$ in which the basic elements in shapes all have the same weight.

Shapes in an algebra $W_{ij}$ are made up of basic elements and weights, so that each shape is a finite but possibly empty set of ordered pairs $(e, u)$. These contain a basic element, $e$, of the algebra $U_{ij}$, and a nonzero weight, $u$. Basic elements of equal weight are maximal, and no two basic elements whatever their weights have an embedded element in common. Thus, for example, the three shapes



in $W_{12}$ that are spatially the same are given by sets of sixteen, twelve, and eight ordered pairs apiece, depending on how weights are assigned.

One aspect of shapes deserves a little more attention. Zero is excluded as a weight. This restriction is enforced for three main reasons. The first is to satisfy the normal expectation that basic elements need some extra property to be seen, the second is to keep basic elements and weights the same in the sense that neither one can be empty in shapes, and the third is to allow for basic elements to be subtracted completely from shapes in the difference operation defined below.

A part relation $\leqslant_W$, and operations of sum $+_W$ and difference $-_W$ are defined in every algebra $W_{ij}$, so that shapes can be compared and combined. The definitions are framed mainly in terms of the part relation $\leqslant_U$ and the operations of sum $+_U$ and difference $-_U$ in the algebra $U_{ij}$, and the composite operation of product $\cdot_U$ in which two shapes $A$ and $B$ in $U_{ij}$ are combined in this way: $A \cdot_U B = A -_U (A -_U B)$. An algorithmic account using the reduction rules for $+_U$ and $-_U$ is thereby readily available whenever it is needed.

*Notation*

Some simple notational conventions and other devices are useful to start. Shapes are special kinds of sets. As a result, shapes can always occur in expressions as sets, and sets of the appropriate kind can occur in expressions as shapes. For example, $x \in A$ indicates that $x$ is a maximal element or an ordered pair in a shape $A$, and $A \cdot_U \{x\}$ is the shape formed in the product of $A$ and the shape given by the set $\{x\}$. For any set of basic elements $\{x\}$, $\sigma\{x\}$ is the set of maximal elements produced by applying the reduction rules for $+_U$. The shape $A^{*}$ is then defined in this way: $A^{*} = \sigma\{e | (e, u) \in A\}$. Further, the shape $\{x\}^{*}$ is obtained in the same fashion from any set of ordered pairs. The set $\sigma\{x\}$ forms a shape by extending the use of the reduction rules for $+_U$ to ordered pairs, whenever no two ordered pairs in the set $\{x\}$ contain basic elements that have an embedded element in common. A new ordered pair $(g, w)$ is obtained from the ordered pairs $(e, u)$ and $(f, v)$ if two additional conditions are satisfied: first, the basic elements $e$ and $f$ are embedded in a common basic element, and they share parts of their boundaries, and, second, the weights $u$ and $v$ are equal. This occurs for ordered pairs in which basic elements are lines in one way

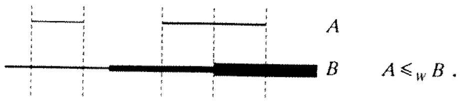
$$\frac{(e, u) \;\vdots\; (f, v)}{(g, w)} \qquad u = v = w\,.$$

Both for basic elements and for ordered pairs, the set $\{x\}$ is a shape just in case $\{x\}$ and $\sigma\{x\}$ are identical.

*The part relation*

Now, consider the part relation $\leqslant_W$. Suppose that $A$ and $B$ are shapes in an algebra $W_{ij}$, and that $(e, u)$ and $(f, v)$ are ordered pairs. Then, $A \leqslant_W B$ whenever both $A^{*} \leqslant_U B^{*}$, and if $(e, u) \in A$ and $(f, v) \in B$, so that the shape $\{e\} \cdot_U \{f\}$ is nonempty (or, equivalently, $e$ and $f$ share an embedded element), then $u \leqslant v$.

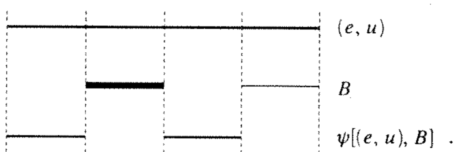
This is illustrated clearly for shapes made up of lines and weights in the algebra $W_{12}$,




The definition of shapes in an algebra $W_{ij}$ and the definition of the part relation have some important implications. In particular, a shape $A$ does not have properties—spatial ones formed by basic elements alone, or other ones formed by weights—by having them as parts. Neither nonempty parts of the shape $A^*$ nor weights related to weights in $A$ are parts of $A$.

*Sum and difference*
Sum $+_W$ and difference $-_W$ are defined with equal facility but with a little less brevity. Roughly speaking, the shape $A +_W B$ produced by combining two shapes $A$ and $B$ in an algebra $W_{ij}$ is formed by dividing the basic elements in $A$ and $B$ into separate pieces to assign weights. These new basic elements are distinguished according to whether they contribute to the shape $A^* -_U B^*$, $B^* -_U A^*$, or $A^* \cdot_U B^*$. These shapes are separate from one another, and exhaust the sum $A^* +_U B^*$. The weights assigned to the basic elements that contribute to $A^* -_U B^*$ and $B^* -_U A^*$ are taken directly from $A$ and $B$, but the weights assigned to the basic elements that contribute to $A^* \cdot_U B^*$ are produced in sums. In the same fashion, the shape $A -_W B$ is formed by dividing the basic elements in $A$ into pieces that contribute to the shapes $A^* -_U B^*$ and $A^* \cdot_U B^*$. The weights assigned to the basic elements that contribute to $A^* \cdot_U B^*$ are produced in differences. However, weights must be more than zero for basic elements to contribute to $A -_W B$. The exact details of both definitions have the complexity and structure of a straightforward textbook problem.

*Assigning weights to $A^* -_U B^*$* If $(e, u)$ is an ordered pair and $B$ is a shape in an algebra $W_{ij}$, then the set $\psi[(e, u), B]$ contains the ordered pairs $(x, u)$, where $x \in \{e\} -_U B^*$. This is easy to see in the algebra $W_{12}$,




Further, if $A$ is a nonempty shape in $W_{ij}$, then the set $\Psi(A, B)$ is given by

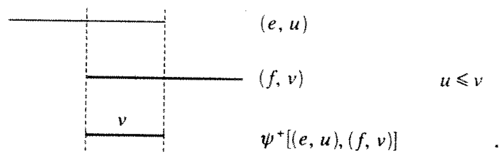
$$\Psi(A, B) = \bigcup_{x \in A} \psi(x, B) .$$

And if $A$ is empty, then $\Psi(A, B) = \varnothing$. Either way, $\Psi(A, B)$ is always a shape in $W_{ij}$.

To make sure of this, consider the consequences if the set $\Psi(A, B)$ is not a shape. The empty set is a shape, so $\Psi(A, B)$ is nonempty. Also, a set containing a single basic element is a shape in $U_{ij}$, and $U_{ij}$ is closed under $-_U$, so the set $\psi(x, B)$ is a shape in $W_{ij}$ for every ordered pair $x$. As a result, $\Psi(A, B)$ must contain an ordered pair $(e', u)$ from a set $\psi[(e, u), B]$ and an ordered pair $(f', v)$ from a different set $\psi[(f, v), B]$, where either the weights $u$ and $v$ are equal and the basic elements $e'$ and $f'$ are not maximal, or $e'$ and $f'$ have an embedded element in common. But this is then the case for the ordered pairs $(e, u)$ and $(f, v)$, because $e'$ and $f'$ are embedded in the basic elements $e$ and $f$, respectively. The condition that $(e, u)$ and $(f, v)$ belong to the shape $A$ is thereby violated.

It is also easy to see that a basic element in an ordered pair in the set $\Psi(A, B)$ is embedded in a basic element in exactly one ordered pair in the shape $A$, and that a weight is assigned in terms of this relationship. Further, it is clear that the shapes $\Psi(A, B)^*$ and $A^* -_U B^*$ are identical.

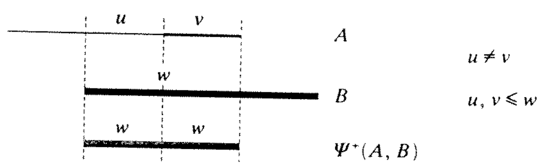
*Assigning weights to $A^* \cdot_U B^*$ in a sum* The same approach works for assigning weights to the basic elements that contribute to the shape $A^* \cdot_U B^*$. Let $(e, u)$ and $(f, v)$ be ordered pairs. Then, for the operation $+_W$, the set $\psi^+[(e, u), (f, v)]$ contains the ordered pairs $(x, u+v)$, where $x \in \{e\} \cdot_U \{f\}$,




Such sets combine to define the set $\Psi^+(A, B)$ given by

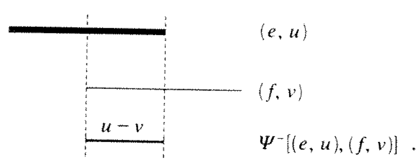
$$\Psi^+(A, B) = \bigcup_{\substack{x \in A \\ y \in B}} \psi^+(x, y),$$

if $A$ and $B$ are both nonempty, and by $\Psi^+(A, B) = \varnothing$, otherwise. Simple reasoning of the kind used previously readily shows that no two ordered pairs in $\Psi^+(A, B)$ contain basic elements that have an embedded element in common. However, the set need not be a shape as this example shows,




In the definition of the set $\Psi(A, B)$, weights are taken directly from ordered pairs, but here weights are obtained from sums. Equal weights produced in different sums may be assigned to basic elements that are not maximal when the ordered pairs in $\Psi^+(A, B)$ are formed by dividing basic elements from $A$ and $B$. For every ordered pair $(g, w)$ in $\Psi^+(A, B)$, there are uniquely corresponding ordered pairs $(e, u)$ and $(f, v)$ in $A$ and $B$, where $g$ is embedded in $e$ and $f$. This relationship fixes the weight $w = u + v$. The shapes $\Psi^+(A, B)^*$ and $A^* \cdot_U B^*$ are identical.

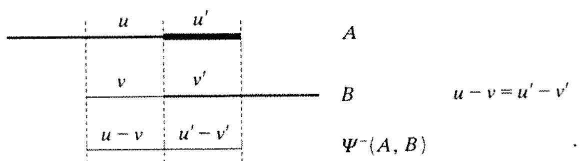
*Assigning weights to $A^* \cdot_U B^*$ in a difference* Finally, for the operation $-_W$, the set $\psi^-[(e, u), (f, v)]$ contains the ordered pairs $(x, u-v)$, where $x \in \{e\} \cdot_U \{f\}$, and $u - v$ is not zero,




These sets combine to define the set $\Psi^-(A, B)$ given by

$$\Psi^-(A, B) = \bigcup_{\substack{x \in A \\ y \in B}} \psi^-(x, y),$$

if $A$ and $B$ are both nonempty, and by $\Psi^{-}(A, B) = \varnothing$ otherwise. No two ordered pairs in $\Psi^{-}(A, B)$ contain basic elements that have an embedded element in common. But a shape is not always formed

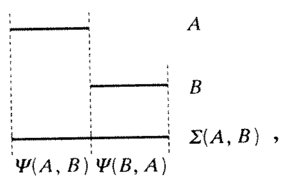



Because some weights may be zero, $\Psi^{-}(A, B)^{*} \leqslant_{U} A^{*} \cdot_{U} B^{*}$.

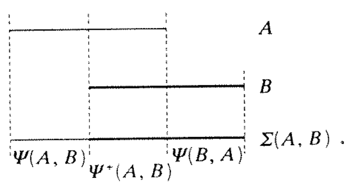
*The definition of sum* Definitions of sum $+_{W}$ and difference $-_{W}$ follow without difficulty. For any two shapes $A$ and $B$ in an algebra $W_{ij}$, the sum $A +_{W} B$ is formed from three sets of ordered pairs: $\Psi(A, B)$, $\Psi(B, A)$, and $\Psi^{+}(A, B)$. These are combined to obtain the set $\Sigma(A, B)$

$$\Sigma(A, B) = \Psi(A, B) \cup \Psi(B, A) \cup \Psi^{+}(A, B) .$$

The basic elements in the ordered pairs in $\Sigma(A, B)$ exhaust the basic elements in the ordered pairs in $A$ and $B$. And because the shapes $A^{*} -_{U} B^{*}$, $B^{*} -_{U} A^{*}$, and $A^{*} \cdot_{U} B^{*}$ corresponding to the sets $\Psi(A, B)$, $\Psi(B, A)$, and $\Psi^{+}(A, B)$ are separate from one another, no two basic elements from $\Sigma(A, B)$ have an embedded element in common. The set $\Sigma(A, B)$, however, is not a shape if $\Sigma(A, B)$ contains ordered pairs $(e, u)$ and $(f, v)$, where $u = v$, and the basic elements $e$ and $f$ are not maximal. This is the case when the set $\Psi^{+}(A, B)$ is not a shape, but may result in other ways, too, either because $A$ and B contain ordered pairs of the necessary kind,




or because such ordered pairs are produced as the sets combined to make $\Sigma(A, B)$ are defined,




The sum $A +_{W} B$ is thus given by

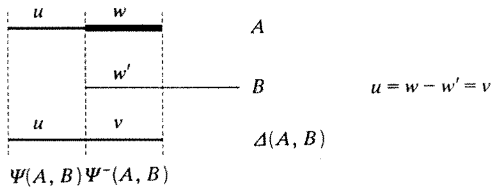
$$A +_{W} B = \sigma\Sigma(A, B) ,$$

so that basic elements of equal weight are maximal.

*The definition of difference* The difference $A -_{W} B$ is formed from two sets of ordered pairs: $\Psi(A, B)$ and $\Psi^{-}(A, B)$. These are combined to obtain the set $\Delta(A, B)$

$$\Delta(A, B) = \Psi(A, B) \cup \Psi^{-}(A, B) .$$

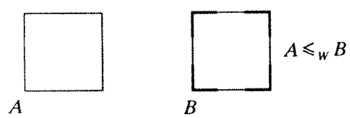
The set $\Delta(A, B)$ is not always a shape. The set $\Psi^{-}(A, B)$ may not be a shape, or two ordered pairs $(e, u)$ and $(f, v)$ in $\Psi(A, B)$ and $\Psi^{-}(A, B)$ may be such that the weight $v$ having been produced in a difference is now equal to the weight $u$ taken from $A$, and the basic elements $e$ and $f$ are not maximal,



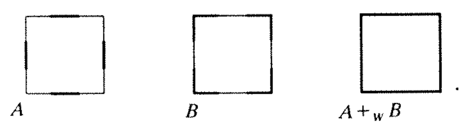

The difference $A -_W B$ is given by

$$A -_W B = \sigma\Delta(A, B).$$

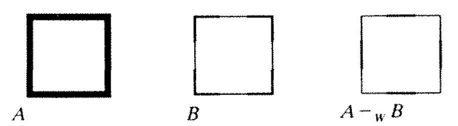
*Discussion*
There are four things to notice about the definitions of the part relation $\leqslant_W$, and the operations of sum $+_W$ and difference $-_W$ in the algebras $W_{ij}$. One: the definitions are framed with ease without loss to the original idea of shapes. The elaboration of this idea in an algebra $U_{ij}$ in terms of the embedding relation, maximal elements, and reduction rules makes this possible in the corresponding algebra $W_{ij}$. Every shape in $W_{ij}$ is identified uniquely by a set with definite members that are independent of one another, but the shape is not thereby limited by this analysis. Two shapes $A$ and $B$ may be related, so that $A \leqslant_W B$, even if they are given by nonempty, disjoint sets,




These sets need not be subsets of the set that forms the sum $A +_W B$,




And the set that forms the difference $A -_W B$ may not be a subset of $A$




Shapes made up of basic elements, whether or not they include weights, come undivided and unanalyzed. They may be described again and again by decomposing them into parts in any way that use demands.

Two: the definitions are meant especially for shapes made up of lines or planes or solids in which every maximal element has infinitely many basic elements embedded in it. The definitions, however, work equally well for shapes made up of points in which every maximal element is atomic, without the possibility of embedding additional basic elements that are different from the maximal element itself. In fact, in this case, there is room for simplification: every set involved in the definitions is a shape. These are the only alternatives for shapes in the algebras $W_{ij}$, but the alternatives intersect elegantly to allow for new kinds of shapes in which maximal elements are built up by combining basic elements that are atomic. This has direct application in computer implementations of the algebras $W_{ij}$, where finiteness limits embedding.

Three: the definitions provide an open framework to form algebras containing shapes made up of basic elements and weights of different kinds. For example, every algebra $V_{ij}$ is isomorphic to another algebra in which labels are treated as weights. Simply represent each individual label $a$ by a singleton set $\{a\}$, let $\leqslant$, $+$, and $-$ be subset, union, and relative complement, and associate nonempty sets with basic elements in ordered pairs. Of course, there is ample opportunity for more complicated algebras, too. Weights may consist of labels and numbers that work together with basic elements, or be other complex objects like vectors and matrices, either given directly or encoded by numbers. Weights may merely distinguish formal attributes, or accord with anything from physical properties like mass on the one hand, to intentional properties like function on the other. (In this richly augmented form, shapes made up of basic elements and weights answer to the traditional Vitruvian categories: physical properties being included in firmness, intentional properties in commodity, and spatial properties in delight.) Weights may be interpreted in any way at all, so long as appropriate definitions of the relation $\leqslant$, and the operations $+$ and $-$ can be found. It may be the case that $\leqslant$, $+$, and $-$ are related in some fashion, or that they are independent of one another. And $\leqslant$, $+$, and $-$ may enjoy algebraic niceties like commutativity for $+$, or be without special distinction.

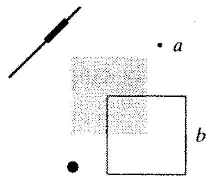
And four: the definitions develop the idea that weights interact in overlapping pieces of basic elements, without additional influence in other places. This is intuitively appealing in art and logic, where colors are mixed in paintings, and propositions are manipulated in Venn diagrams. Nonetheless, there are more ways to think about the relationship between basic elements and weights. One alternative for basic elements of dimension greater than zero is to treat weights in terms of density or some other kind of distribution over maximal elements. A shape then contains ordered pairs in which weights are associated with basic elements that are maximal with respect to one another. And shapes of this kind can be compared and combined in a part relation and in operations of sum and difference involving $\leqslant_U$, $+_U$, and $-_U$.

*Transformations*

It has been tacitly assumed that weights are invariant under the transformations in every algebra $W_{ij}$. This assumption corresponds to everyday practice, but is not crucial. If the assumption is respected, then for a transformation $t$ and a shape $A$ in $W_{ij}$, the shape $t(A)$ is given by the set of ordered pairs $\{[t(e), u]/(e, u) \in A\}$. Otherwise, the shape $t(A)$ depends on the set $\{[t(e), t(u)]/(e, u) \in A$, and $t(u) \neq 0\}$. It is easy to see that this set need not be a shape. The transformation $t$ may be such that, for two ordered pairs $(e, u)$ and $(f, v)$ in $A$, the weights $t(u)$ and $t(v)$ are equal, and the basic elements $t(e)$ and $t(f)$ are not maximal. In this case, the shape $t(A)$ is given by the set $\sigma\{[t(e), t(u)]/(e, u) \in A$, and $t(u) \neq 0\}$.

**Algebras in combination**

Once the algebras $U_{ij}$, $V_{ij}$, and $W_{ij}$ have been defined, they can be combined in some very simple ways to obtain other algebras that are useful in practice. These new algebras contain compound shapes in which basic elements of various kinds, labels, and weights are mixed,




Compound shapes may be formed in unions of the sets that form shapes from different algebras, with the understanding that basic elements and ordered pairs interact if they are the same kind, and are independent otherwise. Or, compound shapes with multiple shapes as components may be formed when algebras are combined in Cartesian products.

**Shape grammars**

Every algebra $U_{ij}$ provides for a diversity of algebras containing shapes made up of basic elements and other things. In each new algebra, the main devices of $U_{ij}$—the part relation $\leqslant_U$, the operations of sum $+_U$ and difference $-_U$, and the transformations—that allow for computations to be defined in shape grammars are maintained uniformly. This makes the use of shape grammars possible without modifying the shape grammar formalism.

All shape grammars depend on the same computational mechanism. In the algebras $U_{ij}$, $V_{ij}$, and $W_{ij}$, and in the algebras formed by combining them, shape grammars contain rules that are followed to carry out computations with shapes. Rules are defined with shapes, and apply recursively first to a given initial shape and then to shapes produced from shapes to determine a series of shapes that forms a computation.

In exact detail, a rule $A \rightarrow B$ establishes a relationship between two shapes $A$ and $B$. The rule applies to another shape $C$ whenever there is a transformation $t$, such that $t(A) \leqslant_X C$. A new shape is then produced according to the formula $[C -_X t(A)] +_X t(B)$. The shapes $A$, $B$, and $C$ are taken from the algebra in which the rule is defined; they may be individual shapes in $U_{ij}$, $V_{ij}$, or $W_{ij}$, or, more generally, they may be compound shapes formed when these algebras are combined. The part relation $\leqslant_X$, and the operations of sum $+_X$ and difference $-_X$ also depend on the algebra in which the rule is defined.

Schemata are sometimes used to extend these devices. A shape schema $A(x)$ is a finite but possibly empty set of variables $x$ that describes a family of shapes. If $A(x)$ is empty, then a shape is given automatically. Otherwise, a function $F$ assigns values to the variables that depend on the algebra in which $A(x)$ is defined. In an algebra $U_{ij}$, $V_{ij}$, or $W_{ij}$, values are basic elements or ordered pairs of a single kind. But in a combination of these algebras containing compound shapes, values may be basic elements and ordered pairs of various kinds. The function $F$ is usually required to satisfy certain other conditions, too, that constrain values or relate them in some way.

In an algebra $W_{ij}$, the values assigned to the variables in a schema $A(x)$ by a function $F$ are ordered pairs containing basic elements and weights. The set $F[A(x)]$, however, may not be a shape. A simple procedure is needed for this purpose. The shape $A$ obtained from the set $F[A(x)]$ is defined in this way.

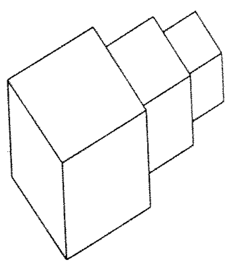
*Let the series $(e_1, u_1) \ldots (e_n, u_n)$ enumerate the ordered pairs in $F[A(x)]$. Also, let the set $A(0)$ be empty. Then, the set $A(i+1)$ is such that $A(i+1) = A(i) +_W \{(e_{i+1}, u_{i+1})\}$. The set $A(n)$ forms the shape $A$.*

It is easy to see that the set $A(n)$ is always a shape: the empty set is a shape, every set containing a single ordered pair is a shape, and $W_{ij}$ is closed under $+_W$. Further, the enumeration of ordered pairs in the set $F[A(x)]$ can be given in any way desired, because $+_W$ is associative and commutative in $W_{ij}$. If this were not so, then the schema $A(x)$ would be given properly by a finite but possibly empty series of variables $x$.

Schemata are also used to define families of objects in Cartesian products, either families of compound shapes or families of rules in shape grammars. For example, any two shape schemata $A(x)$ and $B(x)$—with or without shared variables—that describe shapes in the same algebra can be joined in a rule schema $A(x) \rightarrow B(x)$. A function $F$ assigns values to all of the variables in this schema, so that any required conditions are satisfied. And once the sets defined in this way are used to obtain shapes, as in the procedure for an algebra $W_{ij}$, a rule is formed that may be applied in the manner established above. In effect, this allows for shapes and their relationships to vary within rules, and extends the transformations under which rules apply.

**An example**

It remains to be shown how all of this works in a simple example. Consider rows like this one



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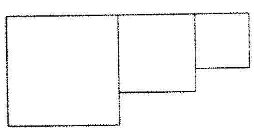
that are formed by joining cubes that decrease in size geometrically at rate $r$. In the algebra $U_{12}$, a row is described by a plan,



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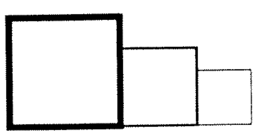
Likewise in the algebra $W_{12}$, a row has an elevation in which line weights decrease as the edges of cubes recede



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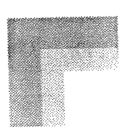
And in the algebra $W_{22}$, a row is described from the side by an arrangement of shaded areas that decrease in weight as the faces of cubes advance,



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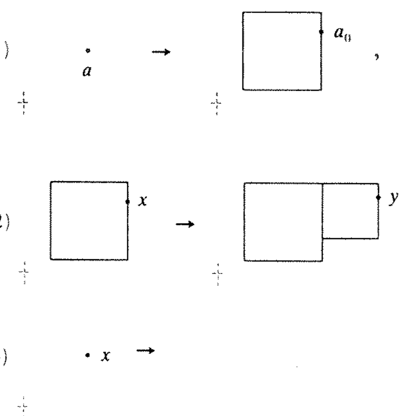
Shape grammars are defined easily to produce either plans, elevations, or arrangements of shaded areas for rows made up of any number of cubes. And these shape grammars can be combined to define a new shape grammar to produce corresponding plans, elevations, and arrangements of shaded areas concurrently.

Schemata are given to define rules in shape grammars, and are presented graphically in the usual way. Variables are used in an ad hoc fashion to range over labels or weights only, with basic elements shown explicitly. Whenever compound shapes are taken from Cartesian products, their components need not be shown separately. Empty components are not indicated at all.

A shape grammar for compound shapes formed by combining the algebras $U_{12}$ and $V_{02}$ is used to obtain plans for rows. The grammar is the familiar kind that carries out computations with lines and labeled points. It contains three schemata to define rules:



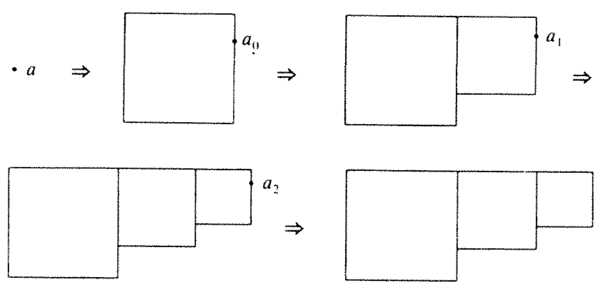

(1) ,

(2) For $i \geqslant 0$: $x = a_i$, $y = a_{i+1}$ ,

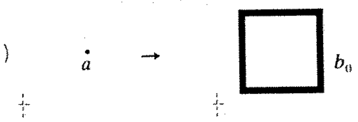
(3) For $i \geqslant 0$: $x = a_i$ .

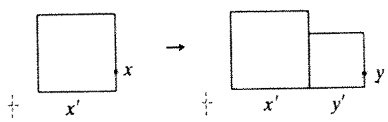
The first schema defines the same rule to start all computations. The rule determines a square with a point labeled by $a_0$ coincident with one of its edges. The second schema provides the rules that are used to continue computations. Each joins a square to a square, and replaces a point labeled by $a_i$ on an edge of the larger square with a point labeled by $a_{i+1}$ on an edge of the smaller square. And the third schema defines the rules needed to complete computations. Each subtracts a point labeled by $a_i$.

The plan produced in this computation beginning with a single point labeled by $a$ describes the row formed by joining three cubes,




Elevations are obtained with comparable ease using compound shapes formed when the algebras $V_{02}$ and $W_{12}$ are combined. The shape grammar defined for this purpose contains three schemata,

(1) 

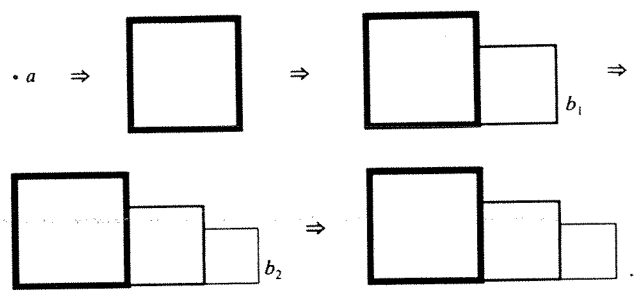

(2) 


For $i \geqslant 0$:
$x = b_i$
$x' = kr^i$
$y = b_{i+1}$
$y' = kr^{i+1}$,

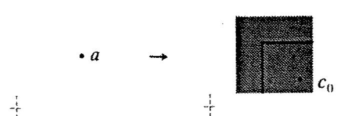
(3) 

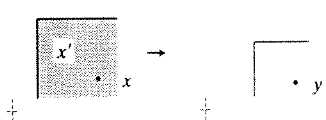

For $i \geqslant 0$:
$x = b_i$ .

These work in the same way as the schemata in the shape grammar for plans. Only now are new labels introduced and weights given to lines. In the first schema, weight is a constant $k$, but in the second schema weights vary according to the labels $b_i$ and $b_{i+1}$ given as values to the variables $x$ and $y$. The row made from three cubes has the elevation produced in this computation beginning with a single point labeled by $a$,

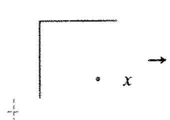



Finally, a shape grammar for compound shapes formed by combining the algebras $U_{12}$, $V_{02}$, and $W_{22}$ is used to obtain arrangements of shaded areas. The grammar contains three schemata,

(1) 


(2) 

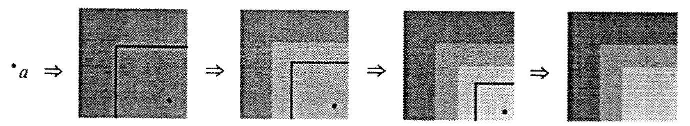

For $i \geqslant 0$:
$x = c_i$
$x' = kr^i(1-r)$
$y = c_{i+1}$ ,

(3) 


For $i \geqslant 0$:
$x = c_i$ .

The first schema defines the same rule to start all computations. The rule determines a square plane of constant weight $k$, with a smaller square marked by two lines in the lower-right corner of the plane and a point labeled by $c_0$ inside the smaller square. The second schema provides rules to continue computations. Each rule does two things: first, it subtracts a square plane with a weight determined by the label $c_i$, and the two lines given to mark the square, and, second, it replaces these liens and the point labeled by $c_i$ with two lines to mark another square, and a point labeled by $c_{i+1}$. The two lines that mark the original square and the point labeled by $c_i$ keep the rule from applying under infinitely many transformations. The third schema defines the rules needed to complete computations. Each subtracts two lines and a point labeled by $c_i$.

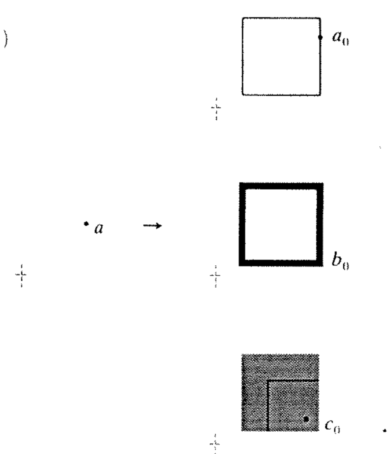
The arrangement of shaded areas produced in this computation beginning with a single point labeled by $a$ describes the row made from three cubes,



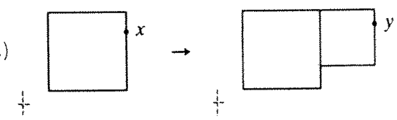

Just a little tinkering is needed to combine these shape grammars for descriptions of rows formed by joining cubes. A new shape grammar is defined that produces corresponding descriptions concurrently by performing separate computations for plans, elevations, and arrangements of shaded areas in parallel. The new shape grammar does its work with rules that are defined by five schemata. Two of the schemata—the one used to start computations and the one used to complete them—coordinate the schemata in the original shape grammars, so that the use of any one of these schemata necessitates the correlative use of certain others. Thus, a change in a description of one kind results in appropriate changes in the descriptions of the other two kinds. A row of cubes may be conceived in terms of different descriptions at different times, without loss to the underlying relationship among descriptions.

The first schema in the new shape grammar combines three schemata, the first from each of the original shape grammars,

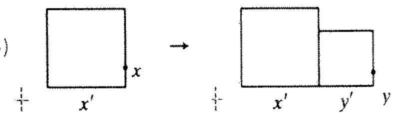
(1)



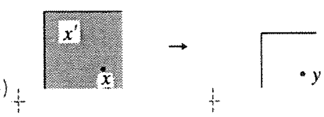

The single rule defined by this schema starts all computations. The rule determines squares for corresponding descriptions, and introduces labels and weights. The rules defined by the next three schemata elaborate descriptions as cubes are joined to make rows longer and longer. The three schemata are taken directly from the original shape grammars,

(2) 


For $i \geqslant 0$:
$x = a_i$
$y = a_i + 1$,

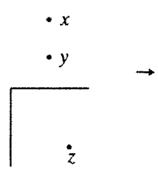
(3) 


For $i \geqslant 0$:
$x = b_i$
$x' = kr^i$
$y = b_{i+1}$
$y' = kr^{i+1}$,

(4) 


For $i \geqslant 0$:
$x = c_i$
$x' = kr^i(1-r)$
$y = c_{i+1}$ .

Finally, the fifth schema combines the third schemata in the original shape grammars,

(5) 


For $i \geqslant 0$:
$x = a_i$
$y = b_i$
$z = c_i$

The labels $a_i$, $b_i$, and $c_i$ are given as values to the variables $x$, $y$, and $z$ associated with the three points in the left side of the schema that have no fixed geometrical relationship. These labels occur together only if corresponding descriptions have been produced. As a result, the schema provides rules to end computations in which rows have been described consistently.

The algebra in which this shape grammar is defined has yet to be given explicitly. The reason is simple: the grammar can be defined equivalently in different algebras. For example, in the Cartesian product of the algebras $U_{12}$, $V_{02}$, $W_{12}$, and $W_{22}$, the plan, elevation, and arrangement of shaded areas produced in the computation in figure 1 describe the row formed by joining three cubes. Shapes in the algebras $U_{12}$, $V_{02}$, $W_{12}$, and $W_{22}$ are given separately, without mixing components or their parts to coincide with the computations given previously. This shows exactly how the computation proceeds in each algebra. Notice especially that the shapes in $U_{12}$ contribute to the formation of both plans and arrangements of shaded areas, and that the shapes in $V_{02}$ contribute to the formation of these descriptions and elevations, too. This kind of sharing is economical, but can be confusing or impractical. If the grammar is defined in the Cartesian product of the algebras $U_{12}$, $V_{02}$, $V_{02}$, $W_{12}$, $U_{12}$, $V_{02}$, and $W_{22}$ in the obvious way, then no ambiguity is possible. Plans are produced in the first two components, elevations in the next two, and arrangements of shaded areas in the final three. And if perfect fastidiousness is desired, then the algebras containing these individual descriptions can be formed to obtain compound shapes in unions of shapes in $U_{12}$ and $V_{02}$, $V_{02}$ and $W_{12}$, and $U_{12}$, $V_{02}$, and $W_{22}$.

This example shows how shape grammars can be used in algebras to define relations among descriptions of different kinds, so that the members of these relations are designs. The beauty of the shape grammar approach is twofold. On the one hand, it allows for descriptions to be framed with shapes made up of points, lines, planes, solids, labels, weights, and other things as needed. And on the other hand, it allows for multiple descriptions to be combined in designs. The interactions in practice among the things in shapes and among the descriptions in designs may be complicated, many, and varied. But the complexity of practice is not beyond the simple, formal techniques enumerated in this paper. Shapes, descriptions, and designs, and the facility to compute with them are to be had with shape grammars defined in the algebras $U_{ij}$ and in the algebras obtained from them.

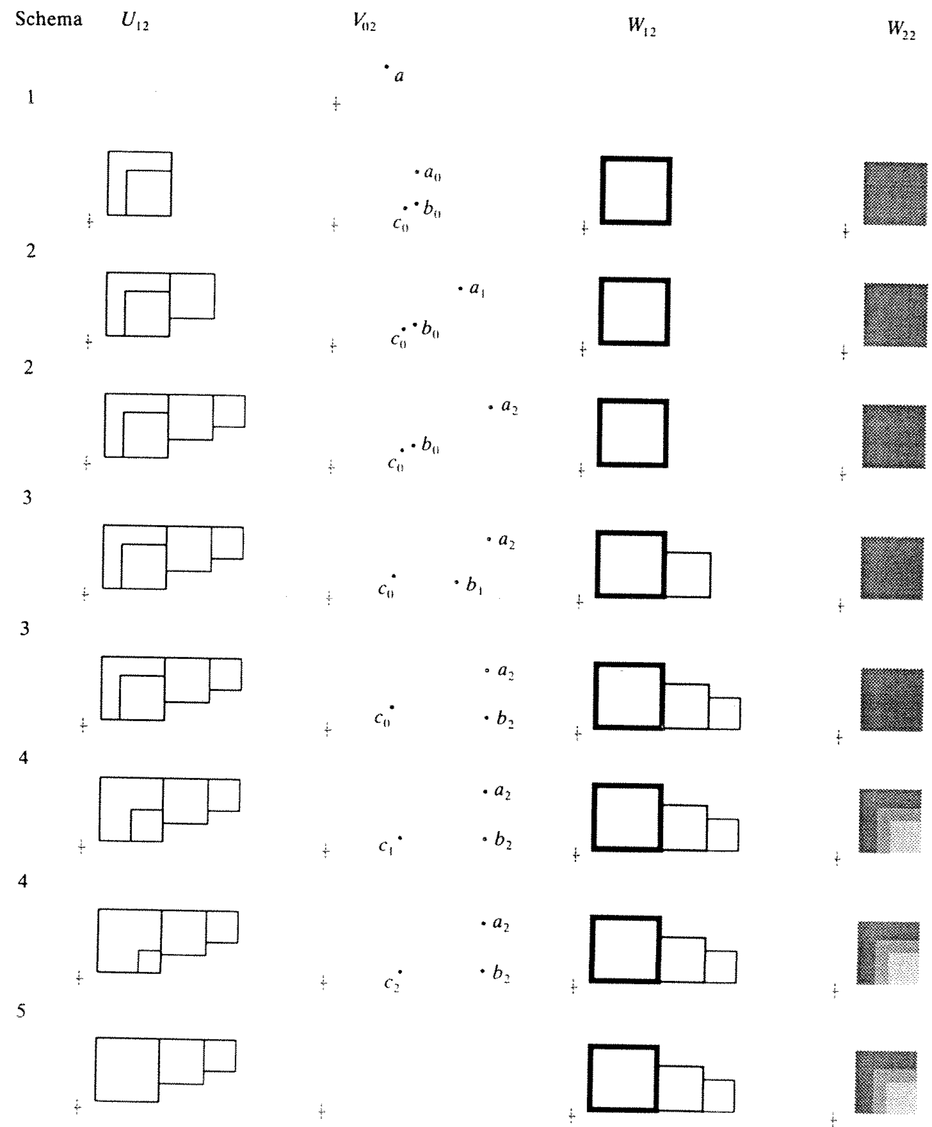



**Figure 1.** A computation in the Cartesian product of the algebras $U_{12}$, $V_{02}$, $W_{12}$, and $W_{22}$.

## Background

The ideas presented in this paper have their roots in shape grammar theory. Stiny (1991) describes the algebras of shapes $U_{ij}$, provides a current discussion of the embedding relation, maximal elements, and reduction rules, and examines several important ways rules work in shape grammars to define computations in the algebras $U_{ij}$ and in their combinations. Stiny and Gips (1972) in the original paper on shape grammars consider the use of properties with shapes, and Knight (1989) explores this theme independently in an alternative fashion. Both of these approaches are subsumed in the development of the algebras $W_{ij}$. Stiny (1990) introduces the idea that designs are members of relations among descriptions of different kinds.

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