Analyzing Fixed Income Portfolios

1 Duration and Convexity

As bond prices depend on the yield curve, to evaluate any portfolio, it is imperative that we know how the value of the portfolio changes when there are changes in the yield curve. There are two measures that are commonly used to summarize exposure to changes in the yield curve: *Duration* and *Convexity*.

If the price of a bond is P and the yield on a bond is y, then the *modified duration*, D^* is defined as

$$D^* = -\frac{1}{P}\frac{dP}{dy}. \tag{1}$$

Thus, the modified duration is a measure of the sensitivity of the price of a bond to small changes in its yield. In particular, if the duration of a bond is higher, then the bond is more sensitive to changes in its yield.

Intuitively, it makes sense to consider an older measure of sensitivity: the *Macaulay Duration*. This is the weighted average time to maturity of the cash flows on a bond or portfolio, where the time to maturity of each cash flow is weighted by its present value divided by the present value of the bond. Define the *Macaulay Duration* as D. Suppose c_t is the coupon payment (so the final coupon payment includes the principal), y is the annualized yield and if there is semi–annual compounding, then the price of a bond, P, is

$$P = \sum_{t=1}^{T} \frac{c_t}{(1+y/2)^t} \tag{2}$$

define

$$\alpha_t = \frac{1}{P}\frac{c_t}{(1+y/2)^t} \tag{3}$$

then,

$$D = \sum_{t=1}^{T} \alpha_t t \tag{4}$$

Notice[1] that the modified duration is just the Macualay duration divided by $(1+y/2)$. So,

$$D^* = \frac{D}{(1+y/2)} \tag{5}$$

As the relationship between a bond's price and it's yield is negative, the modified duration is a positive number. It is important to recognize that duration depends on the yield. That is, if the yield changes then so will the duration. Also, the higher the coupon, the lower the duration. The extreme case is a zero coupon bond, whose Macaulay duration is just equal to its maturity. Finally, duration also depends on time: the duration of any portfolio changes over time.

Another notion that is useful to summarize changes in a bond price in response to changes in its yield, is the *Convexity* of a bond, C which is just:

$$C = \frac{1}{P}\frac{d^2P}{(dy)^2} \tag{6}$$

The number is sometimes quoted divided by 100, as it is so large. You can typically tell from the magnitude what convention is being used.

Convexity is an attractive property in a bond. If you have two bonds with the same value and duration and if interest rates change (either increase or decrease) then the more convex bond will be worth more!

You can calculate convexity by considering any bond as a portfolio of zeros. Recall for a zero coupon bond maturing in t years,

$$P = \frac{100}{(1+y/2)^{2t}} \tag{7}$$

[1]You can verify this yourself by taking the derivative.

SO,

$$\frac{dP}{dy} = \frac{-100t}{(1+y/2)^{2t+1}} \tag{8}$$

$$= \frac{-t}{1+y/2}\frac{100}{(1+y/2)^{2t}} \tag{9}$$

SO,

$$\frac{d^2P}{dy^2} = 100t((2t+1)/2)(1+y/2)^{2t+2} \tag{10}$$

$$= \frac{t(2t+1)/2}{(1+y/2)^2}\frac{100}{(1+y/2)^{2t}}, \tag{11}$$

SO

$$C = \frac{t(t+1/2)}{(1+y/2)^2} \tag{12}$$

Calculation Check: So, if you have a 5 year zero coupon bond with a yield of 6%, the Macaulay duration is just 5 years. The modified duration is:

$$\frac{5}{(1+0.06/2)} = 4.854$$

The convexity is just:

$$\frac{5(5.5)}{(1.03)^2} = 25.9,$$

or 0.259 if you are quoting convexity as a number divided by 100.

To calculate the convexity of a coupon bond, just treat it as a portfolio of zeros. In general, computing these measures for portfolios is easy, once we have computed them for each bond in the portfolio. The portfolio measure is just the weighted average (using the portfolio weights) of the measures on each bond.

Suppose $(w_1, w_2, \ldots w_N)$ are the weights on N bonds in the portfolio, then:

$$D_p = w_1D_1 + w_2D_2 + \ldots + w_ND_N \tag{13}$$

$$D_p^* = w_1D_1^* + w_2D_2^* + \ldots + w_ND_N^* \tag{14}$$

$$C_p = w_1C_1 + w_2C_2 + \ldots + w_NC_N \tag{15}$$

2 Using Duration and Convexity to describe changes in Bond Value

You can use duration and convexity to approximate a change in the value of a portfolio if there is a change in the yield. To do so, take a Taylor's series expansion of the bond price. Recall, the Taylor's series approximation of a function, $f(x)$ around the point (x_0) is:

$$f(x) \approx f(x_0) + \frac{\partial f(x_0)}{\partial x}[x - x_0] + \frac{\partial^2 f(x)}{2(\partial x)^2}[x - x_0]^2 + \ldots h.o.t. \tag{16}$$

(h.o.t. stands for higher order terms. The idea is that these terms are so small that for all intents and purposes you can ignore them.) In our context, the function that we are approximating is the bond price. In what follows, the coupon payment is c and the compounding is semi–annual :

$$P(y) \approx P(y_0) + \frac{\partial P(y_0)}{\partial y}[y - y_0] + \frac{\partial^2 P(y)}{2(\partial y)^2}[y - y_0]^2 + \ldots + h.o.t. \tag{17}$$

If we are only considering small changes, then y is close to y_0, so call the difference $y - y_0$, Δy . If we call $P(y) - P(y_0)$, $\Delta P(y_0)$ then using our definitions of modified duration and convexity:

$$\Delta P(y_0) \approx -D^* P \Delta y + PC(\Delta y)^2 \tag{18}$$

Notice that this expression is valid for small changes in the yield, and that is obtains at an instant in time (i.e. I have held time constant).

3 Portfolio Immunization

Portfolio Managers want to make sure that the value of the fixed income portfolio does not change if there is a change in yield — that is, they do not want to take interest rate bets, either because they want to protect the net worth of their portfolios or because they are worried about fluctuations in the future values of their portfolios (e.g. insurance companies who have know obligations in the future). The first type of immunization is called *net worth immunization*, while the second is called *target date*

immunization. There is also an active management immunization technique called *contingent immunization* that is a combination of net worth immunization and active management.

We know how the value of a bond changes when there is a change in the yield:

$$\Delta P(y_0) \approx -D^* P \Delta y + PC(\Delta y)^2 \tag{19}$$

If $\Delta P(y_0)$=0, then the portfolio does not change when there are small changes in interest rates and hence the yield. To immunize an obligation, set the present value of the obligation equal to the present value of the asset and make sure that the convexity of the asset is greater than the convexity of the obligation. There are a number of different algorithms that can be solved to pick out the best bonds to immunize a particular position.

1. Maximize the convexity of the asset, with the constraint that the durations of the assets and the liabilities are equal and the present values of the two are equal.

2. Minimize the transaction costs (both explicit and opportunity cost) subject to the assets and the liability having the same duration and the asset being more convex that the liability.

3. Maximize the duration and value weighted yield of the portfolio, subject to the duration of the liability equal to the duration of the asset and the convexity of the asset being greater than or equal to the convexity of the obligation.

So, you are constructing a portfolio, with weights: $w_1, w_2, \ldots w_n$, where

$$w_1 = \frac{\text{value of Bond 1}}{\text{Total Portfolio Value}}$$

From the previous handout, *Duration and Convexity*, we know that duration and convexity measures can simply be combined in a portfolio. If $\widetilde{D}$ is the target duration and $\widetilde{V}$ is the target present value, then 1) above becomes:

$$\begin{aligned} \max_{\{w_1,\ldots w_n\}} \quad & \textstyle\sum_{i=1}^n C_i w_1 \\ & \textstyle\sum_{i=1}^n D_i^\star = \widetilde{D} \\ & \textstyle\sum_{i=1}^n V_i = \widetilde{V} \end{aligned} \tag{20}$$

2) is more difficult to formalize explicitly because transaction costs vary across firms. Typically these will depend on the number of bonds that you buy. 3) is seen a lot, but has absolutely no redeeming features.

In general if you are trying to immunize a portfolio, that is, eliminate exposure to interest rate risk make sure that the duration of your assets matches that of your liabilities. Further, the convexity of your assets should be greater than the convexity of your liabilities.

4 Factor Yield Models

Up to this point, we have asked how to protect the value of a portfolio from changes in the yield. Implicitly what we were considering was parallel shifts in the term structure and we tried to determine the effect of those changes on the value of our portfolio and to pick portfolios so there would be no change in value if there was a shift in the term structure (i.e. zero duration portfolios). Now we change focus a bit and hypothesize that the yield on each bond is generated by different factors.

The sorts of factors can be ones that depend on the yield curve such as:

- the level of the yield curve
- the steepness of the yield curve
- the curvature of the yield curve.

Factors could also depend on tax and liquidity effects such as:

- the coupon rate
- the issue size
- for T–bills if they are on the run or seasoned.

There could also be factors due to sector quality spreads such as:

- sectors such as industrials, telephone etc.
- Quality, especially credit quality rating such as AAA or AA.

The yield factor models are similar in structure to the factor models that we saw in equities. In general if there are K factors, $F_1, F_2 \ldots F_K$ and factor loadings $\beta_1, \ldots \beta_K$, then:

$$y_i = A_i + \beta_{i1}F_1 + \beta_{i2}F_2 + \ldots \beta_{iK}F_K. \tag{21}$$

In this case, y_i is the yield on bond i, and β_{i1} is the factor loading on bond i of factor 1.

The simplest example is a one factor model:

$$y_i = A_1 + \beta_i F \tag{22}$$

It immediately follows that

$$\Delta y_i = \beta_i \Delta F \tag{23}$$

You can view the previous types of immunization as special cases where the β of each bond is set to 1, and the factor is a parallel shift in the term structure. The factor approach is then, a more general formulation. If we combine our formulation with the approximation of the change in price of a bond when its yield changes we obtain:

$$\Delta P_1 \approx -P_1 D_1 \beta_1 \Delta F + \frac{C_1 B_1}{2}\beta_1^2 (\Delta F)^2, \tag{24}$$

where β_1 is the factor exposure of bond 1 to the factor F and C_1 and D_1 are the convexity and duration of bond 1 respectively. [2]

The approximate change in value with a change in the factor applies to a particular bond but it also applies to bonds within a portfolio, that is:

$$\sum_{i=1}^{N} P_i \approx -[\sum_{i=1}^{N} P_i D_i \beta_i]\Delta F + \frac{1}{2}\sum_{i=1}^{N} C_i P_i \beta_i^2 \Delta F^2 \tag{25}$$

[2]One way to interpret the Approximation of the change in value of the bond price is to think of the factors affecting the spread between the treasury yield curve and the yield of the actual bond that we are considering. Changes in the factor cause changes in the spreads and so one can justify using the approximation in this context. A more brute force way of thinking about it is to recognize that this is an approximation the only question is how accurate it is.

Now immunizing a portfolio becomes a matter of immunizing either against changes in F — ΔF or larger changes in F, ΔF and ΔF^2. In the first case, set the factor adjusted duration to zero: that is

$$\sum_{i=1}^{N} P_i \beta_i D_i = 0, \tag{26}$$

while in the second case it involves setting both the factor adjusted duration to zero and the factor adjusted convexity:

$$\sum_{i=1}^{N} C_i P_i \frac{1}{2} \beta_i^2 = 0 \tag{27}$$

Factor immunization strategies like this can be used to completely eliminate interest rate risk in net worth immunization, or in the context of active management, to control the overall risk of a position while taking a bet on (for example) the relative movement of long vs. short rates. Factor models are also useful in identifying how the fixed income and equity parts of a portfolio could move together.

For those of you who enjoy nomenclature, the strategy of buying one zero is known as a *bullet*. If you buy a short duration bond and a long duration bond in order to end up with a portfolio with a duration somewhere in between, then you have constructed a *barbell* or as it is sometimes called a *dumbbell*. The strategy of selling a medium duration bond and buying a barbell with the same duration is known as a *butterfly*.