Fixed Income Management

1 Compounding

We will use semi–annual compounding. Sometimes, you will see references to continuous compounding. Semi–annual compounding is practically used while continuous is more convenient mathematically . It is important to recognize that the two methods do not give the same results. Qualitatively they are the same, but quantitatively, they will be different.

Suppose you have \$1 and you invest it for a year. If the annual interest rate is r, then at the end of the year you will get $(1 + r)$. Suppose that the interest is compounded twice annually, then for the first half of the year you earn $(1 + r/2)$, on which interest is earned in the second half of the year. So, at the end of the year you have earned $(1 + r/2)^2$. Generally, you can compound an arbitrary number of times, say m in which case you earn \$ $(1 + r/m)^m$. So, if you invest a larger sum, say A, then if it is invested for t years, and compounded m times you would earn \$ $A(1 + r/m)^{mt}$. You can rewrite the future value of your investment as a function of the number of times which you compound: $V(m)$. If you define $w = m/r$, then you can rewrite:

$$\begin{aligned} V(m) &= A[1 + \frac{1}{w}]^{wrt} \\ lim_{m\to\infty} V(m) &= lim_{w\to\infty} A[1 + \tfrac{1}{w}]^{wrt} \\ &= Ae^{rt}. \end{aligned}$$

So, if you invest A today at a rate r compounded continuously for t years it will be worth V. Or,

$$V = Ae^{rt}.$$

Clearly, the present value of V in t years time is just:

$$A = Ve^{-rt}.$$

Hence, if you have a one year bond that will pay \$40 in 6 months and \$1040 at the end of the year (8% coupon bond) if the annualized 6 month discount rate is 10%, then with continuous compounding the price of the bond is:

$$40e^{-0.1(0.5)} + 1040e^{-0.1(1)}.$$

With semi–annual compounding the price of the bond is:

$$\frac{40}{1+0.05} + \frac{1040}{(1.05)^2}$$

2 T–Bills, T–Notes and T–Bonds

T–Bills are short term government debt instruments and are auctioned off with 3 month, 6 month or 1 year maturities. T– Notes are slightly longer term and are auctioned off for 2 year, 3 year, 5 year and 10 year maturities. T–Bonds are those government debt instruments that are auctioned off with 30 years to maturity. [1]

The price or yield of these securities are determined by auction. After the initial auction these securities are traded on a secondary market. Securities that have just been issues are known as *on the run* securities.

T–Bills are quoted as bank discount yields or simply discount yields. If F is the face value of the security, and P is the price, and there are n days until maturity, then the bank discount yield is

$$D = (1 - \frac{P}{F})\frac{360}{n}.$$

This expression can be used to determine transaction prices. The return on the investment is the *bond equivalent yield*. In this case,

$$y = (\frac{F}{P} - 1)(\frac{365}{n})$$

Notice that the return that you get depends on the type of compounding that is being used.

The quoted price of a *treasury bond* is expressed as a percent of principal value with fractions of a percent expressed in 32nds, 64ths or 128ths. For example, the price 98:24 is $98 + \frac{24}{32}$ for \$100 of par value. The invoice price of a T-bond is its quoted price plus accrued interest on the next coupon. For example, consider a 10%

[1] In the UK the short term government debt: 3 month, 6 month or 1 year is known as T–Bills, while debt up to 5 years are known as short gilts, debt from 5 to 15 years is known as medium gilts and debt greater than 15 years is known as long gilts.

bond which pays a 5% coupon on May 15 and November 15 each year. Suppose that a transaction for this bond settles on September 15. There are 123 days between the last coupon payment (May 15) and September 15 and a total of 184 days between May 15 and November 15. Therefore, the accrued interest to September 15 is 123/184 and

$$\frac{123}{184}5\%$$

would be added onto the quoted price.

3 Bond Yields

Characteristics of a bond are usefully summarized by the *yield to maturity*. That is, the discount rate that makes the cash payments equal to the price. Yield to maturity (out there) is calculated using semi-annual compounding.

Let P be the price of a bond at time t, that matures in T period (for 2 years, T=4) and pays a coupon c_τ semi–annually at each time period τ. The face value of the bond is F. Then, the price is:

$$P = \sum_{\tau=1}^{T} \frac{c_\tau}{(1+(y/2))^\tau} + \frac{F}{(1+(y/2))^T},$$

with yield to maturity y or in the case of continuous compounding:

$$P = \sum_{\tau=1}^{T} e^{-y\tau} c_\tau + F e^{-Ty}$$

The notion of yield to maturity (YTM) sometimes misses important aspects of a Bond's value.

1. YTM ignores Reinvestment Risk
2. It often is not the same thing as the holding period return.
3. If there are complex bond covenants then often the YTM is not a good measure of return.
4. If there is there is a possibility of default then the YTM can be different from the expected yield.

4 The Yield Curve

We've defined the YTM in terms of the price of the bond and the cash flows of the bond. We know that the price of a bond is the present discounted value of the cash flows that accrue to the owner of the bond. The big question is, where do the discount rates come from? To figure those out, go to the treasury market and uncover the **treasury yield curve**. This is the same as the **term structure of interest rates**. That is, for each investment horizon the term structure of interest rates tells you how much per period interest you get if you invest for that length of time.

In order to uncover the yield curve, go to the bond market and find the prices of zero coupon bonds. These are bonds that have no coupon payment and are a commitment to pay the face value at maturity. The yield to maturity on a zero coupon bond is called the **spot rate** that prevails today for a period corresponding to the maturity of the zero. So, for example if the price of a 5 year zero is P_5, then

$$P_5 = \frac{100}{(1 + y/2)^{10}}$$

and

$$y = 2(\frac{100}{P_5})^{1/10} - 1.$$

The yield curve can also be uncovered from coupon bonds. Suppose D_1 is the price today of a zero that pays 1 dollar one period ahead and D_2 is the price today of a zero that pays \$1 in two periods and so on, then the price of a bond at time t that has coupon payments per period c_i and a face value of F_i, can be expressed as:

$$P_i = D_1 c_i + D_2 c_i + \ldots (c_i + F_i) D_T$$

So, if you collect data on T bonds, then you can form a matrix equation:

$$P = CD$$

where P is a $T \times 1$ vector of Bond prices, where the i the row is the price of bond i, P_i .C is a $T \times T$ matrix of coupon payments where the i the row are the coupon payments that accrue to bond i. D is a $T \times 1$ vector of discount factors where the i row is the price today of \$ 1 at time i. One can solve this equation for the discount factors:

$$D = C^{-1} P$$

and hence calculate the discount rates.

It is important to recognize that the yield curve gives us today's price of money in the future. The yield curve can and does change day by day.

An upward sloping yield curve is referred to as the *normal yield* curve because that shape has been observed most often historically. The downward sloping yield curve is referred to as an *inverted yield curve* and has been observed when short rates are extremely high. Once you know the yield curve, you can price claims to cash flows in the future.

One can uncover implied forward rates from the yield curve. The *forward rate* between times t and $t+k$ is $f_{t,t+k}$. This is the amount of interest you earn if you commit today to invest between (for example 6 months in the future and one year). You can calculate the forward rate by supposing that if you buy a t period zero and then reinvest at time t for k periods, that should give you the same future value for $t+k$ periods directly. So, with semi– annual compounding:

$$(1 + r_t/2)^t(1 + f_{t,t+k}/2)^k = (1 + r_{t+k}/2)^{t+k}.$$

Notice that the forward interest rate is not necessarily the rate that actually will prevail in the future. It is merely the rate that you can lock in today for investing in the future.