Portfolio Mechanics

The purpose of the exercise is to combine random variables (the asset returns) to get the highest return for the lowest possible risk (standard deviation). There are other sensible objectives that one can consider, but we will suppose that traders are only interested in mean and standard deviation.

Suppose that there are N assets and each asset $i = 1, \dots N$, has expected return Er_i and standard deviation σ_i. x_i is the weight of asset i in the portfolio. The covariance between asset i and asset j is $cov(r_i, r_j)$, which is sometimes denoted $\rho_{ij}\sigma_i\sigma_j$. The correlation coefficient between asset i and asset j, ρ_{ij}, is a number between -1 (perfect negative correlation) and +1 (perfect positive correlation). The former will be important when we try to hedge portfolios. If ρ is zero then the assets are uncorrelated.

The expected return on a portfolio, Er_p, made up of n such assets is:

$$Er_p = \sum_{i=1}^{n} x_i Er_i.$$

The variance of such a portfolio, σ_p^2 is:

$$\sigma_p^2 = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} x_i x_j cov(r_i, r_j).$$

The mean–variance efficient set or efficient frontier gives the best combinations of expected returns and standard deviation that you can construct. You can determine the best portfolio for a target expected return, $\bar{r}$ by solving:

$$\min_{\{x_i\}_{i=1}^{n}} \sigma_p^2$$

$$\sum_{i=1}^{n} x_i = 1$$

$$Er_p = \bar{r}$$

The portfolio that has the lowest variance irrespective of return is the global minimum variance portfolio. The minimum variance portfolio has weights which are the solution to

$$\min_{\{x_i\}_{i=1}^n} \sigma_p^2$$
$$\sum_{i=1}^n x_i = 1$$

Suppose there is a risk–free asset, with return r_f, then the *optimal risky portfolio* is found by

$$\max_{\{x_i\}_{i=1}^n} \frac{Er_p - r_f}{\sigma_p}$$
$$s.t. \sum_{i=1}^n x_i = 1$$

All this is neater in matrix form: suppose that V is the variance/covariance matrix, that is, it has typical element $\sigma_{i,j}$ — the entry on the i row and jth column. If there are n assets, this is an $n \times n$ matrix. The weights are represented by an $n \times 1$ vector w. Finally, suppose that e is row vector of length n filled with 1's. Then, if the expected returns are in an $n \times 1$ vector called R, the expected return on the portfolio is:

$$w^T R.$$

The variance of the portfolio is

$$w^T V w$$

and the constraint that the weights must sum to one is represented by $w^T e$.

If you are using EXCEL to solve a portfolio optimization problem, be aware that the command for multiplying matrices (arrays) is MMULT and instead of enter, the command is *control shift enter*. The solver is capable of constrained optimization, but be sure that the solution that it generates is the maximum or minimum not just a local optimum.