
33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2020, Deserno

Problem sheet #12

42. Occupation number fluctuations in the free ideal Fermi- and Bose-gas (5 points)

The occupation number n_α of a single-particle energy eigenstate α is a random number. Its *expectation value* is the Fermi or Bose-distribution, $\langle n_\alpha \rangle = (e^{\beta(\epsilon_\alpha - \mu)} \pm 1)^{-1}$. Prove that its *variance* is given by $\sigma_{n_\alpha}^2 = \langle n_\alpha \rangle (1 \mp \langle n_\alpha \rangle)$.

43. Bose-Einstein condensation in a harmonic trap (10 points)

In 1995 three groups of scientists managed to realize a Bose-Einstein condensate of ultra-dilute atomic gases: Eric Cornell and Carl Wieman at JILA, Wolfgang Ketterle at MIT, and Randy Hulet at Rice. For this achievement, Cornell, Wieman, and Ketterle jointly won the 2001 Nobel Prize in Physics. These scientists held the (bosonic) atoms to be condensed in an optically generated *harmonic trap*—i. e., a potential of the form $V(x, y, z) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$. The particles are still non-interacting, but this setup differs from the “free particles *inside a box*” situation we have studied in class. Here we want to see how this changes the physics of Bose-Einstein condensation.

1. What are the single-particle energy eigenstates of this system? Write down the energy spectrum of the single-particle Hamiltonian and—for later convenience—subtract away the irrelevant ground state energy.
2. As a useful result, which will save the day later, prove that $\frac{1}{e^x - 1} = \sum_{k=1}^{\infty} e^{-kx}$, as long as $x > 0$.
3. Bose-Einstein condensation happens if the total number N' of particles contained in *all excited states* is bounded from above. *Without* using a continuum approximation, find an expression for N' and look for an upper bound by focusing on the “best-case scenario” $\mu = 0$. Why is this “best case”? You will end up with an unwieldy looking sum.
4. Rewrite the sum using the identity you proved in part (2), swap the two sums, sum up the inner one, and expand it for large (!) temperatures (you may use a computer to do the latter). Now show that the leading order is

$$N' = \zeta(3) \left(\frac{k_B T}{\hbar \bar{\omega}} \right)^3 + \mathcal{O}(T^2) \quad \text{with } \bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}. \quad (1)$$

5. What is the Einstein temperature T_E for this case? How is this different from what we found for the gas-in-a-box?
6. In the first successful experiment by Ketterle the harmonic trap was claimed to have frequencies $\omega_x = 4681$ Hz, $\omega_y = 1477$ Hz, and $\omega_z = 2576$ Hz. They found Bose-Einstein condensation at about 2 μ K. How many atoms should they have had in the trap at that point? (Turns out, this expectation indeed agreed with independent measurements!)

44. Bose-Einstein condensation in a harmonic trap—finite size corrections (5 points)

Let us look a bit more closely at a simplified version of problem 43, namely the isotropic case $\omega_x = \omega_y = \omega_z \equiv \omega$.

1. Show that continuing the series expansion that led to the answer for problem 43.4 leads to

$$N' = \zeta(3) \left(\frac{k_B T}{\hbar \omega} \right)^3 + \frac{3}{2} \zeta(2) \left(\frac{k_B T}{\hbar \omega} \right)^2 + \zeta(1) \left(\frac{k_B T}{\hbar \omega} \right) + \dots \quad (2)$$

But alas, this high temperature series expansion should strike you as a terrible disappointment. Why?

2. Boldly ignoring your perfectly reasonable anxieties, and taking only *the first two terms* in Eqn. (2), show that they predict a slight downward shift of the Bose-Einstein transition temperature that vanishes in the limit $N \rightarrow \infty$ and is given by

$$\frac{T_{\text{transition}} - T_E}{T_E} = -\frac{\alpha}{N^{1/3}} \quad \text{with } \alpha = \frac{\zeta(2)}{2\zeta(3)^{2/3}} \approx 0.7275. \quad (3)$$

Hint: There's a cubic equation to be solved. Darn. But for large N it suffices to solve this equation approximately. Hence, look for a solution in the form $x(N) = x(N = \infty) - \varepsilon(N)$ and expand to linear order in ε .

3. For extra credit (3 points): Check whether the additional term improves the prediction of the ground state fraction by *numerically* treating the case $N = 1000$. Plot the “exact” $N_0(T)/N$ together with the approximate answers from problem 43.4 and 44.1. You will need to numerically find $\mu(N, T)$ and use this to calculate the ground state occupancy.