33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2020, Deserno

Problem sheet #11

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40. The black body spectrum, and how we see it *(8 points)*

1. Recall that $j_\omega(T) = \frac{\hbar}{(2\pi c)^2}\frac{\omega^3}{\mathrm{e}^{\beta\hbar\omega}-1}$ is the frequency-resolved power radiated by a black body of temperature T per unit area. Express this function in its alternative wavelength-resolved form, $j_\lambda(T)$. Also, calculate $j(T) = \int_0^\infty \mathrm{d}\lambda\; j_\lambda(T)$. *Hint: The transformation theorem for probabilities also works for densities that are not necessarily normalized to unity!*
2. Find the values ω^* $(= 2\pi f^*)$ and λ^* where j_ω and j_λ have their maximum. *(This will require you to numerically solve two transcendental equations.)* Now calculate $\lambda^* f^*$. Does the result surprise you?
3. We get the *perceived brightness* of light by multiplying its power with the *luminous efficiency function* (spectral sensitivity) $V(\lambda)$ of the human eye, which must be determined experimentally. The *luminous flux density* $\mathscr{F}(T)$ is then
$$\mathscr{F}(T) = 683\,\frac{\mathrm{lm}}{\mathrm{W}} \times \int_0^\infty \mathrm{d}\lambda\; j_\lambda(T)\, V(\lambda) \qquad \text{with} \quad V(\lambda) \approx \mathrm{e}^{-(\lambda-\lambda_{\mathrm{max}})^2/2\delta_\lambda^2} \quad \text{and} \quad \begin{aligned}\lambda_{\mathrm{max}} &= 555\,\mathrm{nm}\\ \delta_\lambda &= 43\,\mathrm{nm}\end{aligned}\;. \tag{1}$$
Plot the *overall luminous efficacy*, $\mathscr{E}_{\mathrm{lm}}(T) = \mathscr{F}(T)/j(T)$, as a function of T! *(You need to do the integral numerically.)*
4. White LEDs reach luminous efficacies in excess of 100 lm/W, with even better ones rapidly coming on the market. How much more energy efficient are such LEDs compared to a typical incandescent light bulb operating at 2900 K?

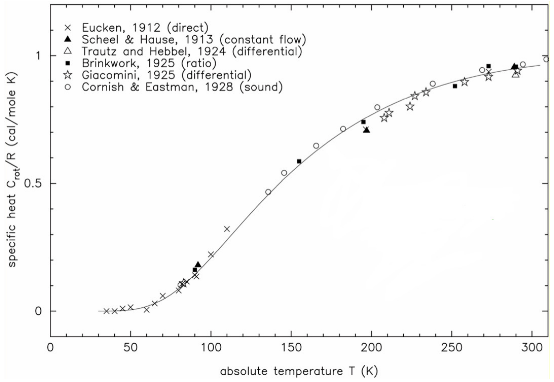
41. The simple and the not-quite-so-simple rigid rotator *(12 points)*

A simple rigid rotator has only rotational energy, and its Hamiltonian is given by $H = \frac{1}{2I}L^2$, where L is the operator of angular momentum and I is the moment of inertia.

1. Remind yourself (and the grader): what are the eigenvalues of L, and what are their degeneracies?
2. Write down the canonical quantum partition function Z of the rigid rotator, as well as its free energy F, energy U, and specific heat c. All these expressions will contain *terrible* sums that you can't do analytically. Don't worry. But you can make things look a wee bit nicer by defining the "rotational temperature" $T_{\mathrm{rot}} = \frac{\hbar^2}{2Ik_{\mathrm{B}}}$.
3. Use your favorite numerical math package to plot $c(T)/k_{\mathrm{B}}$ as a function of T/T_{rot} for $0 \le T/T_{\mathrm{rot}} \le 3$. Do this by *numerically* summing up "sufficiently" many terms. Give an analytical explanation for the limit $T \to \infty$.

Now buckle up: the *nuclear spins* of a hydrogen molecule H_2 can either be parallel, forming a spin-triplet of total spin 1, or antiparallel, forming a spin singlet of total spin 0. The triplet state is called "orthohydrogen", the singlet state is called "parahydrogen". Transitions between them are strongly suppressed by selection rules. Moreover, the fermionic nature of protons imposes restrictions on the rotational states of (ortho/para)hydrogen: the fact that the *spin state* of orthohydrogen is symmetric forces the *wave function* to be antisymmetric, and this means that the eigenvalues ℓ of the angular momentum operator have to be odd. For parahydrogen, this reasoning shows that they have to be even instead.

4. Building on the calculations you have done in parts (1–3), write down a formal expression (containing again yucky sums that you don't need to evaluate) for the (rotational) specific heat c_{para} of *pure parahydrogen*. Consider the restrictions on ℓ!
5. The same for *pure orthohydrogen*, c_{ortho}. Don't forget to count the nuclear triplet degeneracy!
6. Consider now a mixture in which ortho- and parahydrogen are in equilibrium. This requires a "catalyst" to solve the problem that transitions between ortho and para are selection-rule-suppressed. What is c_{eq} of this equilibrium mixture?



7. Without that catalyst, H_2 doesn't easily equilibrate between ortho and para. At high temperature, orthohydrogen is three times as likely as parahydrogen (reflecting the nuclear spin degeneracy), and this is called "normal hydrogen". This ratio stays if one (rapidly) cools hydrogen. Considering that, what now is the heat capacity of "normal hydrogen", c_{normal}?
8. Plot the four theoretically calculated rotational specific heats for $0 \le T/T_{\mathrm{rot}} \le 3.5$. By a <u>careful</u> comparison with the experimentally measured rotational specific heat in the attached figure, *determine the length of the hydrogen-hydrogen bond!* You may consider the mass of a hydrogen atom known, since this is very easy to measure in mass spectrometry.