33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2020, Deserno

Problem sheet #6

20. Inexact differentials, integrating factors, and singular cases *(3 points)*

1. Show that for all $\beta \in \mathbb{R}$, *with one exception*, the differential $đf = \sqrt{\frac{y}{x}}\,\mathrm{d}x - \beta\sqrt{\frac{x}{y}}\,\mathrm{d}y$ is not closed. What is the exception?
2. Show that for all $\beta \in \mathbb{R}$, *with one exception*, there is an $\alpha \in \mathbb{R}$ for which $r(x,y) = \left(\frac{x}{y}\right)^{\alpha}$ becomes an integrating factor of $đf$. What is the exception?

21. Three very simple Legendre transforms *(3 points)*

The Legendre transform $f^{\star}(p)$ of a function $f(x)$ is defined as $\min_x\{f(x) - px\}$, if $f(x)$ is convex, and as $\max_x\{f(x) - px\}$, if $f(x)$ is concave. Calculate the Legendre transform $f^{\star}(p)$ and its derivative $f^{\star\prime}(p) = \partial f^{\star}(p)/\partial p$ for the following functions:

1. $f(x) = \mathrm{e}^x$
2. $f(x) = \log(x)$
3. $f(x) = \cosh(x)$

22. One slightly less simple (but slightly more instructive) Legendre transform *(5 points)*

Same notation and tasks as in problem 21, but now we have $f(x) = \frac{x^2}{1+|x|}$. Plot both $f(x)$ and $f^{\star}(p)$.

Hint: you might want to start by distinguishing positive and negative x.

23. One nontrivial (but enormously instructive) Legendre transform *(9 points)*

Consider the function $f(x)$, which contains a parameter $a \in \mathbb{R}$, and its Legendre transform $f^{\star}(p)$:

$$f(x) \;=\; -\frac{1}{2}ax^2 + \frac{1}{4}x^4 \qquad , \qquad f^{\star}(p) \;=\; \min_x \big\{ f(x) - px \big\} \; .$$

Studying the Legendre transform $f^{\star}(p)$ is nontrivial, because depending on the value of a the function $f(x)$ is *not* everywhere convex. Always keep in mind that the value of a might qualitatively change the results, so be careful about this.

1. Find any minima, maxima, and inflection points of $f(x)$. For which values of a is the function always convex? Sketch $f(x)$ for typical representative cases.
2. In order to actually perform the Legendre transform, you need the equation that links p and x. Find it.
3. The *graph* of $f^{\star}(p)$ is the collection of points $\{p, f^{\star}(p)\}$. Neglecting for a moment the "min" prescription in the Legendre transform, let us consider the collection of points $\{p(x), f(x) - p(x)x\}$, which you could view as a *parametric representation* of the graph of $f^{\star}(p)$ (with x being the parameter). Using your favorite plotting program, provide plots of that graph for representative values of a. What happens when you tune a such that $f(x)$ ceases to be convex? Which bits of the (possibly funny-looking) graph of $f^{\star}(p)$ will survive after applying the "min" in the Legendre transform that we have ignored so far? What therefore happens to the Legendre transform $f^{\star}(p)$ once $f(x)$ deviates locally from convexity?
4. To get $f^{\star}(p)$, we need to solve the equation linking p and x for x. Defining $r^2 = 4a/3$ and $\cos(3\alpha) = 4p/r^3$, show (without using MATHEMATICA or relatives!) that the three solutions $\{x_0, x_1, x_2\}$ can be written as $x_k = r\,\cos\left(\alpha + \frac{2\pi}{3}k\right)$.

   Hint: The trigonometric identity $4\cos^3(A) = 3\cos(A) + \cos(3A)$ should come in handy.
5. Identify the three solutions with the three interesting branches of the Legendre transform which show up once $f(x)$ is no longer convex. Feel free to use your favorite plotting program to do that; no formal proof is required.
6. What is $f^{\star}(0)$? And what is $\lim_{p\to 0^+} f^{\star\prime}(p)$ and $\lim_{p\to 0^-} f^{\star\prime}(p)$?

   Hint: There's no need to suffer through massive differentiation orgies (unless you want to). Simply recall the nice property of Legendre transforms $f^{\star\prime}(p) = -f'^{-1}(p) = -x(p)$.