33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2020, Deserno

Problem sheet #5

16. Rare event statistics is often very counterintuitive *(5 points)*

The daily water height h at some coast line is a random variable. Let's assume it's distributed according to $P_\mu(h) = \frac{4h}{\mu^2}\,\mathrm{e}^{-2h/\mu}$, with $h \in \mathbb{R}^+$. We are now concerned with building a levee that will keep people safe—with an acceptably low risk of failure.

1. Show that $P_\mu(h)$ is normalized and that $\langle h \rangle = \mu$. Plot $P_\mu(h)$ (in suitably chosen units! Think!) as a function of h/μ.
2. Let's assume $\mu = 1\,\mathrm{m}$. How high do we have to build a levee such that the risk of it getting flooded is less than once every 500 years? You will arrive at a transcendental equation for the levee height L, which you have to solve numerically.
3. Assume μ increases by $20\,\mathrm{cm}$ due to global warming. What's now the time scale on which the levee we built is flooded?
4. By how much do we have to increase the height of the levee to get the flooding risk back to once every 500 years?

17. Closed versus exact differentials—the fine difference *(4 points)*

Show that the differential $\text{đ} f = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2+y^2}$ is closed but not exact. Why does this happen?

18. Maximum work from a temperature difference *(6 points)*

Suppose we have two buckets of water with constant heat capacities C_A and C_B, so that the relationship between the change of temperature in bucket i and its change in energy is $\mathrm{d}U_i = C_i\,\mathrm{d}T$. The buckets are initially at temperature $T_{\mathrm{A},0}$ and $T_{\mathrm{B},0}$. We now put an ideal heat engine between these two buckets, depleting that temperature difference to extract mechanical work.

1. What is the final temperature of the water in the two buckets?

   Hint: Start by drawing a diagram of how you connect the buckets and the machine, show your flow of heat and work.
2. What is the maximum amount of work you can extract with such a heat engine from the two buckets?
3. If you just mixed the two buckets of water, instead of using the heat engine, what would be the final water temperature?
4. Is the final temperature in the mixing case higher, lower, or the same as when the heat engine is used? Give *both* a physical argument *and* a mathematical proof of your answer.

   Hint: Your expressions will clear up when you introduce the probability distribution $p_i = C_i/(C_\mathrm{A} + C_\mathrm{B})$.
5. Calculate the change in entropy, ΔS, that occurs when the water is simply mixed together, and *prove* that $\Delta S \geq 0$.

19. Another inequality—just for good measure! *(5 points)*

1. Let $p(x)$ and $p_0(x)$ be two continuous probability densities defined on $\mathbb{R}$. Prove Gibbs' inequality

   $$\int \mathrm{d}x\; p(x) \log\big[p_0(x)\big] \;\leq\; \int \mathrm{d}x\; p(x) \log\big[p(x)\big]\,. \tag{1}$$

   Hint: First prove $\log(x) \leq x - 1$. Now bring the right hand side of (1) to the left, combine, and use the log-inequality.
2. The *von Neumann entropy of a probability density* is defined as the following functional:

   $$S[p] \;=\; -\int \mathrm{d}x\; p(x) \log\big[p(x)\big]\,. \tag{2}$$

   Using the Gibbs inequality, prove the following Theorem: *Among all probability densities of the same variance, the Gaussian has the largest von Neumann entropy.*

   Hint: Start with the Gibbs inequality and choose $p_0(x)$ wisely!