33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2020, Deserno

Problem sheet #4

12. The origin and limitations of classical error propagation *(5 points)*

Consider a collection of random variables $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$, from which we calculate a function of interest, $F(\boldsymbol{X})$. Assume we know all expectation values $\langle X_i \rangle$ and even all covariances $C_{ij} := \text{Cov}(X_i, X_j) = \big\langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \big\rangle$.

1. Taylor-expand F to *second order* around $\langle \boldsymbol{X} \rangle$. Now take the average and show how $\langle F(\boldsymbol{X}) \rangle$ differs from $F(\langle \boldsymbol{X} \rangle)$.
2. For the special case of $n = 1$ and a convex F, show that your result is consistent with Jensen's inequality!
3. The variance of F is given by $\sigma_F^2 = \big\langle [F(\boldsymbol{X}) - \langle F(\boldsymbol{X}) \rangle]^2 \big\rangle$. Simplify this by replacing F with its *first order* Taylor expansion. Show further that *if all X_i are uncorrelated*, you end up with the "standard" formula for error propagation!
   Note: if the random variables X_i are correlated, the more general error propagation formula is not necessarily much more complicated. For inexplicable reasons, this hardly ever seems to be taught. Well, now you know. You're welcome.

13. Generalized geometric and arithmetic mean *(3 points)*

Let $\{x_i\}_{i=1\dots N}$ be a set of N positive real numbers and $\{p_i\}$ be a probability distribution. Prove the following inequality between a *generalized arithmetic mean* and a *generalized geometric mean*:

$$\sum_{i=1}^{N} p_i \, x_i \;\geq\; \prod_{i=1}^{N} x_i^{p_i} \; . \tag{1}$$

14. Pumping gas *(4 points)*

When Alice needs to go to the gas station, she always purchases gasoline for a fixed amount of money. When Bob needs to get gas, he always fills up the whole tank. Considering that gas prices fluctuate, show that these two strategies differ economically! Try to estimate how much better the cheaper strategy is.

Hint: assume that whenever Alice or Bob go to the gas station, the "price per mile", p_i, is a random variable with some unknown distribution. Calculate the total fuel cost of Bob after N visits to the gas station, and the total number of miles Alice reaches after N visits. Then calculate the effective average price per mile after N visits for Alice and Bob. Now remember Jensen.

15. Work done by a moving piston—valuable lessons from kinetic theory *(8 points)*

In class we studied the pressure exerted by an ideal gas onto a hard wall within the framework of kinetic theory. Here we want to extend these thoughts and contemplate what happens if the wall (let's now call it a "piston") moves against the gas.

1. Assume the piston moves towards the ideal gas with a constant velocity u. Using kinetic theory, and contemplating the choice of a clever frame of reference, show that the pressure P_{p}, which the gas exerts onto the piston, is given by

$$P_{\text{p}} = P\left[(1+\tilde{u}^2)\left(1 + \text{erf}\frac{\tilde{u}}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}\,\tilde{u}\,\mathrm{e}^{-\tilde{u}^2/2}\right] = P\left[1 + 2\sqrt{\frac{2}{\pi}}\tilde{u} + \tilde{u}^2 + \mathcal{O}(\tilde{u}^3)\right] , \tag{2}$$

   where P is the pressure of the idea gas, $\overline{v} = 1/\sqrt{\beta m}$ is the root mean square velocity of the particles in one direction, and $\tilde{u} = u/\overline{v}$ is the scaled piston velocity. This yucky expression comes from an integral which you can use MATHEMATICA to solve and expand. I care *much* more about you being able to *explain* carefully, what is the correct integral to start with.
2. Calculate the rate $\Delta E/\Delta t$ at which the piston adds energy to the gas, and show that it vanishes in the limit $u \to 0$.
3. We (usually) do not care *how long* a piston moves, but *by how much*. Calculate the total energy change that happens when a piston compresses the gas by a volume ΔV, while moving at velocity u. Show that in the limit $u \to 0$ this reduces to the well-known expression $\Delta E = P\,\Delta V$ (which—great Scott!—we have thereby revealed to be an approximation).
4. If the piston performs harmonic oscillations with amplitude a and frequency ω, show that to lowest order the time-averaged rate of energy change is $\overline{\Delta E/\Delta t} = W a\omega^2/\sqrt{2\pi}\overline{v}$, where $W = P\,\Delta V$ is the equilibrium work done in one stroke.
5. In real life pistons don't ever move infinitely slowly. But then, we usually pretend that they do. Can we really? Estimate whether $\Delta E = P\,\Delta V$ is a good approximation for the pistons in a car engine, which runs at about 3000 rpm!