33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2020, Deserno

Problem sheet #3

8. The Gamma function *(4 points)*

One of the first "special functions" one learns about in college—after having covered exponentials, logarithms, and trigonometric functions in high school—is probably the Gamma function, $\Gamma(x)$, which is usually defined as the following integral:

$$\Gamma(x) = \int_0^\infty \mathrm{d}t \; t^{x-1} \mathrm{e}^{-t} \qquad x \in \mathbb{R}^+ \qquad (\text{in fact: } x \in \mathbb{C},\ \mathrm{Re}(x) > 0) \; . \tag{1}$$

Prove the recurrence relation $\Gamma(x+1) = x\,\Gamma(x)$ and deduce from it that $\Gamma(N+1) = N!$ for $N \in \mathbb{N}_0$. Also, calculate $\Gamma(\frac{1}{2})$.

9. The Cauchy-Lorentz distribution *(6 points)*

The p-density of the Cauchy-Lorentz distribution $p_{x_0,a}(x)$ with location parameter x_0 and scale parameter a is defined as

$$p_{x_0,a}(x) \;=\; \frac{a/\pi}{a^2 + (x-x_0)^2} \; . \tag{2}$$

1. Show that $p_{x_0,a}(x)$ is normalized. Also show that none of its higher moments exist.
2. For the special case $x_0 = 0$, calculate the characteristic function $\tilde{p}_{0,a}(k)$. *Hint: residue theorem…*
3. Let the random variables X_i be the outcomes of independent measurements of some observable that is Cauchy-Lorentz distributed with $p_{0,a}(x)$. We want to accurately measure that observable, so we define the average $Y_N = \frac{1}{N}\sum_{i=1}^N X_i$ and aim for a large N. What is the p-density of Y_N? Discuss your findings in the context of the Central Limit Theorem!

10. Another frequently occurring distribution function *(6 points)*

Let X be a random variable which has the p-density $P_{\mu,\sigma^2}(x)$, given by

$$P_{\mu,\sigma^2}(x) \;=\; \frac{1}{x\,\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\} \qquad x \in \mathbb{R}^+ \; . \tag{3}$$

1. Show that $P_{\mu,\sigma^2}(x)$ is normalized and that its r^{th} moment is $\langle X^r \rangle = \mathrm{e}^{r\mu + \frac{1}{2}r^2\sigma^2}$ for all $r \in \mathbb{R}$.
   Hint: complete the square in some exponent in some integral after some obvious substitution…
2. If $P(x)$ is a p-density, a number m (not necessarily unique!) such that $\int_{-\infty}^m \mathrm{d}x\, P(x) = \frac{1}{2}$ is called a *median* of that distribution. Show that for $P_{\mu,\sigma^2}(x)$ the median is unique and given by $m = \mathrm{e}^\mu$.
3. If we consider the new random variable $Y = \log X$, show that its p-density is a Gaussian with mean μ and variance σ^2.
4. Plot the distribution function $P_{\mu,\sigma^2}(x)$ for the three cases $\mu = 0$ and $\sigma^2 \in \{\frac{1}{3}, 1, 3\}$ *in the same diagram* (you might want to choose $x \in [0,5]$). Mark the mean and the median in each case. Comment on what you find noteworthy.

11. Jensen's inequality *(4 points)*

Let $f : \mathbb{R} \supset G \to \mathbb{R},\ x \mapsto f(x)$ be a function which is *convex* (some people would say: "convex up") and (for simplicity) differentiable on the connected domain G. As a consequence, for all $x, x_0 \in G$ we have

$$f(x) \;\geq\; f(x_0) + f'(x_0)\,(x - x_0) \; . \tag{4}$$

1. Draw an educationally pristine graph to illustrate that the inequality (4) makes sense as a definition of convexity.
2. Let X be a continuous random variable and $f(x)$ be a convex (differentiable) function. Prove *Jensen's inequality*:

   $$\big\langle f(X) \big\rangle \;\geq\; f\big(\langle X \rangle\big) \; , \tag{5}$$

   where the average is taken over whatever p-density underlies the randomness of X.
   Hint: Begin by choosing a suitable x_0 in the convexity inequality (4).