33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2020, Deserno

Problem sheet #2

4. Fun with the transformation theorem *(6 points)*

Let X and Y be two real random variables, *independently* and *uniformly* chosen from the interval $[0, 1]$.

1. Define the new random variable $Z := Y/X$. What is its probability density $P_Z(z)$?
2. What is the probability that Z rounds to an even integer?
3. What is the probability that Z rounds down ($\longrightarrow$ `''floor''`) to an even integer?

Hints: Zero is even, too. Also, in part 2 and 3, you will encounter infinite series that are well known but slightly tricky. It is sufficient if you Google their value and leave a brief comment why that value is correct.

5. And yet another application of the transformation theorem *(4 points)*

Let X and Y be two independent real random variables which are both distributed according to a Gaussian with mean zero and variance one. Define the new random variable $Z = X/Y$. What is the probability density of Z?

6. Poisson distribution *(4 points)*

Another (discrete!) distribution which one frequently encounters is the so-called "Poisson distribution". It is defined by

$$P_\mu(n) = \frac{\mu^n}{n!}\mathrm{e}^{-\mu} \qquad n \in \mathbb{N}_0\ ,\ \mu \in \mathbb{R}^+\ . \tag{1}$$

Show that $P_\mu(n)$ is properly normalized and calculate its expectation value $\langle n \rangle$ and variance σ_n^2!

Hint: $\langle n^2 \rangle$ is a bit finicky to calculate directly, but $\langle n(n-1) \rangle$ isn't...

7. More on the Poisson distribution *(6 points)*

In problem 5 we encountered the discrete Poisson distribution function $P_\mu(n) = \mu^n \mathrm{e}^{-\mu}/n!$. It is a good model to describe the number n of random events that independently occur in some interval of time, during which the *expected* number if μ.

1. It turns out that one way to think about the Poisson distribution is as follows: consider a Bernoulli process with N trials and success probability p, and imagine the limit in which $N \to \infty$, $p \to 0$, but $Np = \mu = $ const. Show that in this limit the associated binomial distribution function $P_{\text{bin}}(n; N, p)$ converges towards the Poisson distribution $P_\mu(n)$!
2. Check this statement *numerically* by graphically comparing the distribution function $P_{10}(n)$ with several Bernoulli distribution functions of increasingly large N and small p, such that $Np = 10$.
3. And finally: a neat application to the Poisson distribution. A support center receives calls from customers who need help with some product. The calls arrive randomly and independently of each other, but historical data shows that the center receives on average 10 calls per hour. Beyond a certain number of calls in any given hour the support line is overwhelmed and the system collapses, so the center needs to make sure to employ enough operators to handle occasional rushes.
   a) At least how many calls does the support center have to be able to handle within an hour so that the probability of being overwhelmed is less than 0.1%?
   b) Repeat your calculation for three successively bigger call centers that receive on average 20, 50, and 100 calls per hour. Use your findings to argue why large call centers can be run more efficiently than small ones!

   Hint: you will need to calculate these probabilities numerically—there's no (easy) closed expression.