33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 14
Due Friday, Dec. 9

READING: Kreyszig Ch. 4, all sections.

EXERCISES (available as pdf file at www.andrew.cmu.edu/course/33-759):

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. To understand the power series method it is essential to apply it to some examples. A number of simple (and some not so simple) cases are given in the exercises at the end of Secs. 4.1 and 4.2 in Kreyszig. Turn in solutions to No. 10 of Sec. 4.1 and No. 4 of Sec. 4.2, and for each include the following: (i) a general recursion relation for the coefficients; (ii) an explicit solution to this recursion relation, with undetermined constants clearly indicated; (iii) an explicit sum of the resulting power series to give some elementary function or functions; (iv) a check that the function(s) obtained this way actually satisfy the original differential equations.

3. OPTIONAL Find all the singular points in the finite z plane of each of the following linear differential equations. Identify which are regular singular points, and which are not.

$$\begin{aligned}
&\text{(i):} && z^2y'' + z^3y' - (z^2-2)y = 0,\\
&\text{(ii):} && (1-z^2)y'' + (1-z)y' + y = 0,\\
&\text{(iii):} && (\tan z)y'' + (\pi^2 - z^2)y' + (\sin z)y = 0,\\
&\text{(iv):} && z^2y'' + (\tanh z)y' + 3y = 0.
\end{aligned}$$

4. a) Find the singular points of the differential equation

$$(-2 + z^2 + z^3)y'' + 5y' + \sin(\pi z)y = 0.$$

At each singular point find the roots of the indicial equation, and then discuss the form of solutions near this singular point using the discussion in Kreyszig, p. 213 (handed out in class).

b) On the basis of (a), discuss what you might expect to be the radius of convergence of a series solution to the differential equation about $z = 0$. Note that there may be more than one power series solution, so you may want to take account of this in your discussion.

5. Based on Kreyszig Sec. 4.3 Prob. 10.
The generating function for Legendre polynomials is

$$G(u,x) = \frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n.$$

a) Write a computer algebra program (Maple or Mathematica) which expands the generating function $G(u,x)$ out to order u^5, and use it to produce the polynomials $P_n(x)$ for $0 \le n \le 5$. Check that you get the answers on p. 208 of Kreyszig.

b) Use the generating function to show that if $\mathbf{r}_1$ and $\mathbf{r}_2$ two points in space and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, then

$$\frac{1}{r} = \frac{1}{r_2} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r_1}{r_2}\right)^n,$$

where r, r_1, and r_2 are the magnitudes of $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$, respectively, θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, and $r_1 < r_2$, so that the sum converges.

c) Use the generating function to show that $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $P_{2n+1}(0) = 0$, and

$$P_{2n}(0) = (-1)^n \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}\right).$$

d) Use Maple or Mathematica to check numerically whether the Bonnet recursion relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

works when n is *not* an integer. For example, try $n = 1.6$ and various values of x. As well as x in the range $-1 < x < 1$, try some values of $x > 1$ or $x < -1$, and also some complex values of x. Do the same thing for $Q_n(x)$, the second solution to Legendre's equations for a given n, both for integer and non-integer values of n. Describe what you have done and what you conclude in a brief essay (half a page).

6. Both $J_\nu(x)$ and $Y_\nu(x)$, $\nu \geq 0$ fixed, are oscillating functions of $x > 0$, and if they are plotted on the same graph, the zeros of one fall between zeros of the other in the sense that between two successive zeros of one of these functions one always finds a zero of the other.

a) Confirm this behavior for $\nu = 1$ by plotting both functions on the same graph in the range $0 < x < 20$. (Since $Y_1(x)$ diverges as x goes to zero, choose some reasonable value of x at which to begin your plot of this function.)

b) Prove that this behavior is true in general, for any value of $\nu \geq 0$, by using the fact that $J_\nu(x)$ and $Y_\nu(x)$ are independent solutions to Bessel's equation. [Hint. What can you say about the Wronskian? Consider what happens to dJ_ν/dx between two successive zeros of $J_\nu(x)$.]

7. Modified Bessel functions satisfy a differential equation

$$z^2y'' + zy' - (z^2 + \nu^2)y = 0.$$

The solution to this equation denoted by $I_\nu(z)$ is closely related to the solution $J_\nu(z)$ of the ordinary Bessel equation. Assuming that $I_\nu(z)$ is real when $z = x$ is real and positive, how can it be expressed in terms of $J_\nu(z)$? (You can look this up, but it is more useful to work out the connection yourself – you should come up with the right answer within a $\pm$ sign and the location of a branch cut.)

8. Kreyszig Sec. 4.7, problem 7. When you have found the answer, check that

$$\int_1^e p(x)y_m(x)y_n(x)\,dx = 0 \text{ for } m \neq n$$

by explicitly carrying out the integral.

9. Find the first three terms in the Fourier-Bessel expansion of the function $f(x) = x$ on the interval $0 \leq x \leq 1$ as a a series in $J_1(k_{m1}x)$, $m = 1, 2, \ldots$, where $J_1(k_{m1}) = 0$. (See Example 3 on p. 243 of Kreyszig, and set $R = 1$.) The integrals can be evaluated in closed form (in terms of Bessel functions), but you can also evaluate them numerically if you prefer. Make a plot or plots showing the first three partial sums of the series on the interval interval $0 \leq x \leq 1$, and show $f(x)$ on the same plot(s).