33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 13
Due Friday, Dec. 2

READING: Kreyszig: Ch. 1, Secs. 1 2, 3, 5, 9. Chapter 2: take a look at Secs. 1, 2, 3, 7, 8, 9, 10. We will not be going through this material directly in class, but instead covering the same ground using the methods of Ch. 3, with some extensions in the Differential Equations Supplement. Chapter 3: Secs. 1, 2, 3, 6. Differential Equations Supplement, Secs. 1 through 6.

READING AHEAD: Kreyszig Ch. 4, Secs. 1 through 6; Ch. 11, Secs. 1, 2, 3, 5, 10.

EXERCISES (available as pdf file at www.andrew.cmu.edu/course/33-759):

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. Let

$$f(x) = \begin{cases} e^{-\lambda x} & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Find the convolution

$$h(y) = \int_{-\infty}^{\infty} f(x) f(y-x)\, dx,$$

of f with itself, both by direct integration and by Fourier inversion of $(\hat{f}(k))^2$. [Hint: Pay attention to whether y is positive or negative.]

3. You should be able to do exercises 2 through 25 at the end of Kreyszig Sec. 1.3, given the hints he provides. Do not turn anything in.

4. Choose some exercises at random at the end of Kreyszig Sec. 1.5, and make sure you know how to do them. Do not turn anything in.

5. Riley, Hobson and Bence, Exercise 14.4.
Find the values of α and β that make

$$F(x,y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) dx + (xy^\beta + a)\, dy$$

an exact differential. For these values solve $F(x,y) = 0$.

6. Riley, Hobson and Bence, Exercise 14.6. Find an appropriate integrating factor to solve

$$\frac{dy}{dx} = -\frac{2x^2 + y^2 + x}{xy}.$$

7. Kreyszig Sec. 1.9, problems 1 and 2, slightly modified.
Consider the differential equation $x dy/dx = 4y$. Are there solutions to the initial value problem with (i) $y(0) = 1$, (ii) $y(0) = 0$? If solutions exist, are they unique? Discuss how the results are related to Kreyszig's existence and uniqueness theorems on pp. 53 and 54.

8. OPTIONAL. Kreyszig Sec. 3.1, Nos. 7 and 8.

(continued)

9. Relating higher-order linear equations to systems of first-order differential equations.

a) Show that the third-order equation

$$\mathcal{L}y := y^{(3)} + b_2(x)y^{(2)} + b_1(x)y^{(1)} + b_0(x)y = g(x)$$

is equivalent to

$$\mathbf{y}' = A\mathbf{y} + \mathbf{g},$$

where y_1, y_2, y_3 correspond to y, $y^{(1)}$ and $y^{(2)}$; $A(x)$ is a suitable 3×3 matrix, and $\mathbf{g}(x)$ a 3-component column vector. You should work out the entries of $A(x)$ and $\mathbf{g}(x)$ in terms of the quantities in the original equation.

b) Consider the case of constant coefficients $b_0(x) = 2$, $b_1(x) = -3$, $b_2(x) = 0$. Show that the characteristic equation obtained by inserting the ansatz $y = e^{\lambda x}$ in the homogeneous differential equation $\mathcal{L}y = 0$ is (essentially) the same as the characteristic polynomial $\text{Det}[A - \lambda I]$ for the matrix A, and find its roots.

c) Find the general solution to the homogeneous equation $\mathcal{L}y = 0$ considered in (b), and use it to construct a basis of three solutions $\mathbf{y}^j$, $j = 1, 2, 3$, of the homogeneous system $\mathbf{y}' = A\mathbf{y}$. Be sure and check that your $\mathbf{y}^j(x)$ actually are solutions.

d) Find a basis in which the matrix A for the choice of coefficients in (b) is in Jordan block form. (You will find that the results in (c) are helpful in this connection.)

10. Find a homogeneous second order linear differential equation for which the functions

$$y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$$

are two independent solutions. Find the Wronskian $w(x)$ and show that it satisfies the expected differential equation. Does the fact that $w(x)$ vanishes at $x = 0$ contradict the rule that the Wronskian should either be zero for all x or nonzero for all x? Discuss.

11. Kreyszig Sec. 3.3, Nos. 4 and 10.

12. (Based on Kreyszig No. 6 in Problem Set 3.6.) Consider the system of equations

$$\mathbf{y}' = A\mathbf{y} + \mathbf{g} \;\; \text{where} \;\; A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

Remember that the inverse of any 2×2 matrix has the simple form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

a) Find two independent solutions to the homogeneous equation, call them $\mathbf{y}^1(x)$ and $\mathbf{y}^2(x)$, and form the 2×2 matrix

$$\bar{Y}(x) = \left(\mathbf{y}^1(x), \mathbf{y}^2(x)\right).$$

b) Evaluate the Wronskian $w(x) = \text{Det}[\bar{Y}(x)]$, and show that it satisfies the expected differential equation.

c) Obtain $Y(x) := e^{xA}$ from $\bar{Y}(x)$ by finding a constant matrix B such that

$$Y(x) := e^{xA} = \bar{Y}(x)B.$$

Verify directly by differentiating its matrix elements that Y satisfies the equation $Y' = AY$.

d) Find the general solution to the inhomogeneous equation when $g_1(x) = x$, $g_2(x) = -x$ by guessing a particular solution of the form $\mathbf{y} = \mathbf{r} + x\mathbf{s}$, solving appropriate equations for the constant vectors $\mathbf{r}$ and $\mathbf{s}$, and then using this as part of the general solution.

e) Find a particular solution to the same inhomogeneous equation by the method of variation of constants, that is, by using $Y^{-1}(x)$ or $\bar{Y}^{-1}(x)$. Either will work, but one of them is likely to be simpler, so use it rather than the other.

f) Use the method employed in (e) to obtain a particular solution for the case in which $g_1(x) = e^{2x}$ and $g_2(x) = e^{5x}$. Check by differentiation that the resulting $\mathbf{y}$ is a solution of $\mathbf{y}' = A\mathbf{y} + \mathbf{g}$.