33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 12 (Final version)
Due Friday, Nov. 18

READING: Kreyszig, Ch. 10, on Fourier series and transforms, except for Sec. 10.6. (Sec. 10.11 is just a table of transforms.) Fourier Series and Transforms Supplement (FSTS).

EXERCISES (available as pdf file at www.andrew.cmu.edu/course/33-759):

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. Exercises 1 to 18 at the end of Kreyszig Sec. 10.1 are relatively simple. Look at a few of them and make sure you know how to do them. Do not hand in any.

3. a) Transform the following two finite trigonometric series from the a_n, b_n form to the c_n form (find the non-zero c_n).

$$\sin x + \tfrac{1}{3}\sin 3x,$$
$$1 + \cos x + \tfrac{1}{2}\cos 2x$$

b) Transform the following two finite trigonometric series from the c_n form to the a_n, b_n form (find the non-zero a_n and b_n).

$$2 + e^{ix} + \tfrac{1}{2}e^{i2x},$$
$$(1-i)e^{-ix} + 3 + (1+i)e^{ix}$$

4. The exercises in Sec. 2.4 of FSTS are fairly simple; check that you know how to do them. Turn in a solution to 2.10.

5. Kreyszig Sec. 10.3, No. 6. Find the nonzero Fourier coefficients, a_n or b_n, by direct integration and also by using the "differentiation trick", which in this case means you need to differentiate $f(x)$ twice. Make computer plots of the partial sums on separate graphs, over a range that includes two periods, and for comparison show the original $f(x)$ on each plot. (Sketch in $f(x)$ by hand if you have trouble getting the computer to do it.)

6. (This is similar to problems 20 to 25 in Kreyszig Sec. 10.4) Sketch the odd and even periodic extensions of

$$f(x) = \sin x, \quad 0 \le x \le \pi,$$

and then work out the Fourier sine and cosine series.

7. Find the Fourier series for the periodic function $f(x+2\pi) = f(x)$ defined by

$$f(x) = e^{ix/2} \text{ for } -\pi < x < \pi,$$

by evaluating the $\{c_n\}$ using appropriate integrals. Next, use the values you obtained for the $\{c_n\}$ to find the corresponding $\{a_n\}$ and $\{b_n\}$, and relate them to the Fourier series of $u(x)$ and $v(x)$ when one writes $f(x) = u(x) + iv(x)$ as a sum of its real and imaginary parts.

8. Show that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$$

by the following procedure. Find the function $g(x)$ whose average is zero and whose *derivative* is the rectangular wave $f(x)$ of unit amplitude on the interval $-\pi$ to π, and find the Fourier series of $g(x)$ by integrating that for $f(x)$—the latter is given in Kreyszig Sec. 10.2; set $k = 1$. Next apply the Parseval identity to $g(x)$.

9. a) Find the Fourier transforms $\hat{f}(w)$ and $\hat{g}(w)$ of the following two functions:

$$f(x) = \begin{cases} \sin x & \text{for } 0 \le x \le \pi, \\ 0 & \text{elsewhere.} \end{cases} \qquad g(x) = \begin{cases} \cos x & \text{for } -\pi/2 \le x \le \pi/2, \\ 0 & \text{elsewhere.} \end{cases}$$

Verify that $\hat{f}(w)$ and $\hat{g}(w)$ are related in the way one would expect, given that

$$g(x) = f(x + \pi/2).$$

Are $\hat{f}(w)$ and $\hat{g}(w)$ singular or well-behaved at $w = \pm 1$?

b) OPTIONAL. Plot the real and imaginary parts of $\hat{f}(w)$, and $\hat{g}(w)$ over the range $-6 < w < 6$.

10. What properties of its Fourier transform $\hat{f}(k)$ characterize a function $f(x)$ which is both real, $f^*(x) = f(x)$, and odd, $f(-x) = -f(x)$? Same question for a function $f(x)$ which is both pure imaginary and even. Suppose both $\hat{f}(k)$ and $f(x)$ are real: what is special about each function?

11. Let $f(x)$ be defined for $0 \le x \le \infty$, and let $\mathcal{F}_c(f)$ and $\mathcal{F}_s(f)$ be its cosine and sine transforms. Define

$$f_0(x) = \begin{cases} f(x) \text{ for } x \ge 0 \\ 0 \text{ for } x < 0 \end{cases}$$

on the interval $-\infty < x < \infty$.

a) Show that

$$\mathcal{F}(f_0) = \tfrac{1}{2}\left[\mathcal{F}_c(f) - i\mathcal{F}_s(f)\right],$$

where $\mathcal{F}$ is the ordinary Fourier transform.

b) Let $f(x) = e^{-kx}$. Find $\mathcal{F}(f_0)$, and from this deduce both the cosine and the sine transforms of $= e^{-kx}$.