33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 11.
Handed out on Nov. 2
(Not to be turned in)

ANNOUNCEMENT: There will be an hour exam on Wednesday, Nov. 9 at 3:30. It will be in the regular classroom, WEH 7316, which is free at 4:30 as there is no seminar scheduled at that time. This exam will be on the part of the course devoted to probability theory, including Markov chains. You are responsible for assigned material in Kreyszig Ch. 22 Secs. 2-9, along with the part of Ch. 23 Sec. 23.3 devoted to the central limit theorem; the Probability Theory Supplement, including the expanded version of Sec. 4; and the Markov Chains Supplement. The examination will be closed book and closed notes.

READING AHEAD: Kreyszig, Ch. 10, on Fourier series and transforms, except for Sec. 10.6. (Sec. 10.11 is just a table of transforms.)

EXERCISES: *DO NOT TURN IN!*

In reviewing for the hour exam, you may find it helpful to look at the exercises at the ends of Ch. 22 "Chapter Review", in Kreyszig, and the exercises intercalated in the supplementary notes handed out in class. You should be able to answer these with book or notes closed.

Fall 2004 Hour Exam:

1. (45 points) Widgets emerging from a factory assembly line are subject to two quality tests, Test I and Test II, before they are shipped to customers. Any widget that fails Test I is discarded, whereas if it passes Test I it is subjected to Test II. If it passes Test II it is shipped to the customer, but if it fails Test II it is sent to the repair department for adjustments, and Test II is applied again. If it passes Test II this time, it goes to the customer; if not, it is discarded. Let A_1 be the event that a widget passes Test I, F_1 the event that it fails this test; likewise A_2 and F_2 are passing or failing Test II immediately after Test I; A_3 and F_3 are passing or failing Test II following an adjustment. Let π_1 be the probability that a widget passes Test I, π_2 the probability that it passes Test II given that it has just passed Test I, and π_3 the probability that it passes Test II given that it has been adjusted (because it failed Test II the first time).

a) Describe the sample space, preferably using the symbols A_1, F_2, etc. You may abbreviate $A_1 \cap F_2$ to A_1F_2 (or A_1, F_2), etc. You need not include in the sample space elements which have zero probability (never occur). Write down the probability for each element of the sample space in terms of π_1, π_2, π_3, and check that the sum of these probabilities is what you expect.

b) Both π_2 and π_3 are actually conditional probabilities. Express them in this form, i.e., as $\Pr(B \mid C)$, where B and C are expressed in terms of A_1, F_1, etc., and check that this is consistent with the probabilities you assigned to points in the sample space in part (a), by using them to find $\Pr(B \cap C)$ and $\Pr(C)$, and taking the ratio.

c) Consider a widget which is shipped to a customer. What is the probability that it failed Test II the first time and had to be adjusted? To receive credit for this part of the problem you must explain your reasoning. Find a numerical value assuming that $\pi_1 = 0.8$, $\pi_2 = 0.6$, and $\pi_3 = 0.5$.

(IGNORE parts (a), (b), and (c) of problem 2, as we have not taken up moment generating functions.)

2. (55 points) a) How is the moment generating function $M_X(t)$ defined for a random variable X? Write down an expression for $M_X(t)$ in terms of the probability distribution density $\rho(x)$ of the random variable X. Express the expectation $E(X)$ and the variance $\sigma^2 = \mathrm{Var}(X)$ in terms of the moment generating function.

b) Let $\{X_j, 1 \leq j \leq n\}$ be n independent, identically-distributed random variables, each with a mean μ, variance σ^2, and generating function $M_X(t)$. Define

$$S_n = X_1 + X_2 + \cdots X_n.$$

Derive expressions for the mean and variance of S_n in terms of μ and σ^2.

c) Derive a formula relating $M_{S_n}(t)$, the moment generating function of S_n, to $M_X(t)$, and make clear where the *independence* of the X_j enters in your derivation. Then use the formulas relating mean and variance to the generating function, as in (a), to derive expressions for the mean and variance of S_n in terms of μ and σ^2. These should, of course, agree with your results in (c).

d) The central limit theorem says that when n is large the cumulative distribution

$$F_n(s) = \Pr(S_n \leq s)$$

can be approximated in some way using the cumulative probability density $\Phi(y)$ for a normal random variable with zero mean and a variance of 1. Explain the connection. (You are not being asked to derive the central limit theorem, but instead to show that you know how to apply it. The end result might look like $?F_n(?) \approx ?\Phi(?)$.)

F2003 There was no hour exam for probability theory. Part of a question on the final:

A collection of continuous random variables $X_1, X_2, \ldots X_n$ is said to be independent and identically distributed provided the joint probability distribution density is of the form

$$\rho(x_1, x_2, \ldots x_n) = f(x_1)f(x_2)\cdots f(x_n),$$

where on the right side $f(x_j)$ is always the same function, independent of j.

a) Find expressions for the expectation (average) M and variance S^2 of the random variable

$$Y = X_1 + X_2 + \cdots X_n$$

in terms of the properties of $f(x)$. Be explicit and make clear what you are doing. [Hint: Your answer will involve integrals.]

[Parts (b) and (c) have to do with confidence intervals, not part of the F2005 course.]