33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 10. (Final version)
Due Friday, Nov. 4

ANNOUNCEMENT: Hour exam Wed., Nov. 9, 3:30 pm. Probability theory, Markov chains.

READING: Kreyszig Secs. 22.8 and parts of 23.3 relevant to the central limit theorem. Markov Chains Supplement handed out in class.

READING AHEAD: Kreyszig, Ch. 10, on Fourier series and transforms, except for Sec. 10.6. (Sec. 10.11 is just a table of transforms.)

EXERCISES (available as pdf file at www.andrew.cmu.edu/course/33-759):

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. You should be able to do exercise 1-13 at the end of Kreyszig Sec. 22.8. Turn in solutions to Nos. 2 and 12.

3. Example illustrating one of the dangers of the useful-but-dangerous formula

$$\sigma(y)\,dy = \rho(x)\,dx.$$

Suppose that X is uniformly distributed on the interval $[-1,1]$, with probability density

$$\rho(x) = \begin{cases} 1/2 & \text{for } -1 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Y = X^2$, so $y = h(x) = x^2$. Consequently, the probability density for Y is

$$\sigma(y) = \rho(x)\,\frac{dx}{dy} = \frac{\rho(x)}{2\sqrt{y}} = \begin{cases} (4\sqrt{y})^{-1} & \text{for } 0 \le y \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Show that the result is incorrect, find the correct density, and figure out what went wrong with the derivation based on the dangerous formula. [Hint. If you run into any difficulty, think about the case in which X is uniformly distributed on $[0,1]$ rather than $[-1,1]$.]

4. Consider three radioactive nuclei of the same kind, and let the decay times be the three independent random variables T_1, T_2, and T_3, each having an exponential distribution $\rho(t) = \lambda e^{-\lambda t}$ for $t \ge 0$.

a) Find the joint probability distribution density $\bar{\rho}(t_1, t_2, t_3)$ (the bar is used to distinguish this from ρ for a single nucleus), and the joint cumulative distribution $\bar{F}(t_1, t_2, t_3)$.

b) For a fixed $t > 0$, let N_t be the random variable denoting how many nuclei have already decayed by time t. Find the probability distribution $f(N_t)$ and use it to compute the mean and the variance of N_t, all of which will depend upon t as a parameter. Is the t dependence of $f(N_t = 0)$ and $f(N_t = 3)$ reasonable, i.e., what you might have expected on physical grounds? [Hint: If you find this confusing, try first working it out for one rather than three decaying nuclei.]

5. An experiment is being designed to measure the lifetime of a rare isotope that is hard to produce, so one would like to do this using a minimum number n of decays. Use a Gaussian (normal) distribution to estimate what n must be so that the probability is less than 5% that the measured average lifetime differs from the true value by more than 2%. Indicate your reasoning.

6. The transition matrix P_{ij}, $1 \le i, j \le 6$ for a Markov chain is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0.2 & 0.5 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0.6 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \end{pmatrix}$$

a) Identify the different classes, stating what states are in each class, and whether the class is recurrent or transient, periodic with a particular period (state what that is) or aperiodic. [Hint. You may find it useful to draw a graph.]

b) The transition matrix ("little matrix") for a recurrent class is obtained by discarding from P all the rows and columns which do not belong to this class. For each recurrent class identified in (a), find its (little) transition matrix $\bar{P}$, and the corresponding invariant probability distribution $\bar{\pi}$. Find the mean recurrence time of every recurrent state. In which cases will $\bar{P}^n$ tend to a limit as $n \to \infty$, and what is this limit? If there is no limit, can you nonetheless produce one by a suitable averaging? [Hint: See the handout.]

c) Not to be turned in. Show numerically that a suitable average of P^n approaches a limit when n becomes large. Then give this limit an interpretation in terms of the invariant probabilities computed in (b). [Hint. Consider an initial probability distribution $f_j^0 = 1/6$. What will happen to f^n when n becomes large?]

7. Consider a 3-state Markov process with a transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1/3 & 2/3 \end{pmatrix}$$

a) Construct the corresponding graph with directed edges. Identify the classes, indicating their types, and find the "little" matrix associated with the recurrent class.

b) Find the eigenvalues and left eigenvectors of P. (Equivalent to finding the right eigenvectors of the transpose P^{T}.) How are these related to the eigenvalues and eigenvectors of the little matrix? What is the invariant distribution π? Can you give a reason why one of its components should be zero?

c) (You do NOT have to write up this part of the exercise, but you will find it instructive to actually carry it out.) Work out numerical values for the probability distribution (vector) at time n for a few values of n starting with (i) $f^0 = (1, 0, 0)$, (ii) $f^0 = (0, 1, 0)$, (iii) $f^0 = (0, 0, 1)$. It is worth writing a little program to do this. Verify that in each case f^n is tending towards the invariant distribution.

d) Let $d^n = f^n - \pi$, i.e. $d_j^n = f_j^n - \pi_j$, be the difference between betwen the probability distribution at time n and the $n \to \infty$ limit. Discuss the asymptotic behavior of d^n as $n \to \infty$. Show in particular that in the general case d^n will decrease exponentially with a certain exponent related to eigenvalues found in (a), but that if f^0 is suitably restricted, in a manner you should indicate, there is an exponential decrease, of a somewhat different character, again related to what you found in (a). The *amplitudes* of these exponentially decreasing terms is the subject of part (e) below. [Hint. Expand f^0 in terms of the left eigenvectors of P. Also your numerical results in (c) may be helpful.]

e) OPTIONAL. The left eigenvectors of P form a basis. Explain how with help of the dual basis (consisting of right eigenvectors of P) one can write down a closed-form expression for f^n for any n in terms of eigenvalues and coefficients determined using f^0 and this dual basis.

8. Exercise associated with *Markov Chains Supplement*, Sec. 4.

a) Carry out the exercise associated with (4.6).

b) Show that the transition matrix P_{jk} corresponding to the Metropolis algorithm described at the end of Sec. 4 satisfies the detailed balance condition (4.5).