33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 9.
Due Friday, Oct. 28

READING: Kreyszig Secs. 22.2 through 22.7; 22.9. Probability Theory Supplement (handed out in class), Secs. 1 through 8.

READING AHEAD: Kreyszig Secs. 22.8 and parts of 23.3 relevant to the central limit theorem. Markov Chains Supplement (to be handed out in class).

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. Do the following for each of the three statements below. Construct a Venn diagram, and use the Venn diagram to figure out why the implication is true. Then use the axioms of probability theory to construct an argument or proof that the statement is correct. (Axioms: probabilities are nonnegative, are additive for disjoint sets, and the probability of the entire sample space is 1.)

a) $B \subset A$ implies $\Pr(B) \leq \Pr(A)$.

b) $\Pr(A \,|\, B) = 1$ implies $\Pr(A \,|\, B \cap C) = 1$.

c) $\Pr(A \,|\, B) = 1$ and $\Pr(B \,|\, C) = 1$ implies $\Pr(A \,|\, C) = 1$.

3. You should be able to do all of the exercises at the end of Kreyszig Sec. 22.3. Turn in solutions to Nos. 8 and 10, and in each case indicate explicitly the *sample space* involved; in particular, give the total number of elements in the sample space.

4. Combinatorial problems at the end of Kreyszig Sec. 22.4. You should know how to do these (apart from No. 15). Turn in solutions to No. 6 and No. 10. For the latter, consider two seating arrangements distinct if and only if someone has a different neighbor to his right.

5. You should be able to do all the exercises at the end of Kreyszig Sec. 22.5. Turn in solutions to Nos. 6 and 10. In the case of No. 10, identify the sample space and indicate the number of elements it contains.

6. You should be able to do all the exercises at the end of Kreyszig Sec. 22.6, apart from No. 16, which you can ignore. Turn in solutions to Nos. 7, 8, and 9, regarded as one problem.

(continued on other side)

7. Define the cumulative probability function

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x/2 & \text{for } 0 < x < 1/2 \\ x/2 + 1/4 & \text{for } 1/2 < x < 1 \\ 1 & \text{for } 1 < x. \end{cases}$$

a) Sketch $F(x)$ and find $\rho(x) = dF/dx$, including appropriate Dirac deltas.

b) For which values of $a < b$ is it the case that

$$\Pr(a < X < b) < \Pr(a \le X \le b)?$$

For these cases also discuss the values of $\Pr(a < X \le b)$ and $\Pr(a \le X < b)$.

c) Compute $E(X)$ and $E(X^2)$ using the $\rho(x)$ you found in (a), and from these determine σ^2.

d) Let $\xi(F)$ be the inverse function to $F(x)$, i.e., $x = \xi(F(x))$. Write down an explicit formula defining it in different intervals of F (in the same way that $F(x)$ was defined above). Sketch $\xi(F)$ and then use it to calculate $E(X)$ and $E(X^2)$ by carrying out integrals of $\xi(x)$ and $\xi^2(x)$. Show your work. You should get the same answer as in (c).

8. a) For a fair die ($p_s = 1/6$ for $1 \le s \le 6$), let X be the random variable that takes the value 1 if s is even and 0 if s is odd, and $Y = (s-3)^2$. Set up a table showing the joint probability distribution $f(x, y)$ following the example in Kreyszig Table 22.2, p. 1095: Rows labeled by x with values increasing from top to bottom, columns labeled by y with values increasing from left to right, marginal distributions indicated in the bottom row and right-most column. Why is it obvious from the table that X and Y are not statistically independent? Compute the expectation and the variance for both X and Y, and the covariance.

b) Use the same X as in (a), but choose a different random variable Y which takes on at least three different values (with probability greater than zero) so that X and Y are *independent*. Again construct a probability table, compute the the expectation and variance for both X and Y, and check that the covariance is what you expect.

9. You should be able to do exercises 1-13 at the end of Kreyszig Sec. 22.7. Turn in solutions to:

Number 4. Calculate numerical values, to four decimal places, for both the exact (binomial) and approximate (Poisson) distribution, assuming the same mean value μ in both cases, so that you can compare the two. It may be worth writing a little program.

Number 10. Obtain numerical values to four decimal places, and check that the probabilities sum to 1.