33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 8.
Handed out on Oct. 12
(Not to be turned in)

ANNOUNCEMENT: There will be an hour exam on Wednesday, Oct. 19 at 3:30 IN ROOM DH A301D. (Not in WEH 7316!) It will cover the part of the course devoted to linear algebra. This includes Chs. 6 and 7 of Kreyszig, with the exception of Sec. 7.2, and the Linear Algebra Supplement notes handed out in class (10 Oct. 2005 version). The "Linear Algebra Outline" (3 pages) handed out in class lists all of the topics which have been covered, though not necessarily in the same order they were taken up in the lectures.

The examination will be closed book and closed notes.

READING AHEAD: Kreyszig Secs. 22.2 through 22.8; Probability Theory Supplement (handed out in class), Secs. 1 through 7. notes on

EXERCISES: *DO NOT TURN IN!*

In reviewing for the hour exam, you may find it helpful to look at the exercises at the ends of Chs. 6 and 7, "Chapter Review", in Kreyszig, and the exercises intercalated in the Linear Algebra Supplement notes.

Fall 2004 Hour Exam:

1. Each of the following statements is almost, but not quite, correct. In each case find the (or at leaat *a*) correct statement by making a (small) change in the original statement, or perhaps adding a qualification. While doing this, or in addition, indicate what was wrong with the original statement. Use this as an opportunity to demonstrate that you understand the subject.

a) Complex polynomials of a complex variable of degree less than or equal to 3 form a linear vector space of dimension 3, given the usual rules of addition of polynomials and multiplication by a complex scalar.

b) Two eigenvectors of a normal operator are always orthogonal to each other.

c) If $\hat{A}$ and $\hat{B}$ are Hermitian operators, so is $\mu\hat{A} + \nu\hat{B}$.

d) Any square matrix A can be brought to diagonal form D by means of a similarity transformation $D = S^{-1}AS$.

e) An $N \times N$ matrix A with real eigenvalues and N linearly-independent eigenvectors is Hermitian.

f) If two normal operators $\hat{A}$ and $\hat{B}$ commute with each other, and $\hat{A}$ is diagonal in the orthonormal basis $\{\mathbf{e}_j\}$, then so is $\hat{B}$.

g) To determine whether $\hat{A}$ is a normal operator, calculate its matrix A, and let $A^\dagger$ be the matrix which is the complex conjugate of the transpose of A. The operator $\hat{A}$ is normal if and only if $AA^\dagger = A^\dagger A$.

2. Each of the following eigenvalue/eigenvector problems is simple if you know what you are doing. Give a brief indication of your reasoning process in each case.

a) Choose values of a, b, and c such that the following matrix will have real eigenvalues.

$$\begin{pmatrix} 1 & 1+i & 1-i \\ a & 2 & 0 \\ b & c & 1 \end{pmatrix}$$

b) Find the sum and the product of the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ i & 2 & 1-i \\ 3 & 0 & 4 \end{pmatrix}$$

c) Write down *all* of the eigenvectors of the following matrix, and indicate the corresponding eigenvalues.

$$\begin{pmatrix} 1+i & 1 & 0 & 0 \\ 0 & 1+i & 1 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

d) What are the eigenvalues of the following matrix

$$\begin{pmatrix} 1+i & 3 & 4-i \\ 0 & 2 & 3-i \\ 0 & 0 & -2 \end{pmatrix},$$

and can it be diagonalized by means of a similarity transformation?

e) Find conditions on a, b, and c such that the eigenvalues of the following matrix are all of unit modulus, $|\lambda_j| = 1$.

$$\begin{pmatrix} i/\sqrt{2} & a \\ b & c \end{pmatrix}$$

Linear algebra part of Fall 2003 Hour Exam:

1. A linear operator $F : V \to V$ on the complex vector space V has a matrix $\mathbf{F}$ in a particular basis $\{\mathbf{e}^j\}$, and the characteristic polynomial is

$$\mathrm{Det}(\mathbf{F} - \lambda \mathbf{I}) = -\lambda(1.5 - \lambda)^2,$$

where $\mathbf{I}$ is the identity matrix.

a) What is the dimension of the vector space V, and what are the eigenvalues of F? How many eigenvectors are associated with each of the eigenvalues? (Don't forget to give at least a brief explanation in each case.)

b) Is F a unitary operator, or could it be a unitary operator? Is F a Hermitian operator, or could it be a Hermitian operator? If it is Hermitian, is it a positive operator? Discuss.

2. a) Define the rank of a matrix. Find the rank of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ i & 1 & 0 \\ 1 & i & 2 \end{pmatrix}.$$

Make sure (with additional explanation if needed) that your method relates your answer to your original definition of rank.

b) Under what conditions on $\mathbf{b}$ will the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

have some solution $\mathbf{x} = \mathbf{x}_0$?

c) When a solution exists, will it be unique? If your answer is "yes", explain why. If "no", indicate how the other solutions are related to $\mathbf{x}_0$.