33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 7.
Due Friday, Oct. 14

READING: Kreyszig Secs. 6.5, 7.1, 7.3, 7.4, 7.5. Linear Algebra Supplement, Secs. 11 through 17

READING AHEAD: Kreyszig Secs. 22.2 through 22.8. (Also glance through Sec. 22.1 for terminology; you can ignore stem-and-leaf plots and boxplots.)

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. Eigenvalues and eigenvectors of simple matrices. You should know how to find them for the examples given in Kreyszig Sec. 7.1, Nos. 1 to 14. TURN IN SOLUTIONS to No. 12 and No. 14. Also take a look at Kreyszig Sec. 7.3, Nos. 1 to 6, and 7.4, Nos. 5 to 10, but do not turn in solutions.

3. OPTIONAL. DO NOT TURN IN. Which of the following matrices are Hermitian? Which are unitary? For those that are neither Hermitian nor unitary, are there any that are normal?

$$\text{(i)}\ \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \quad \text{(ii)}\ \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ -1-i & 1-i \end{pmatrix} \quad \text{(iii)}\ \begin{pmatrix} 1 & 1+i \\ 1+i & 1 \end{pmatrix}$$

$$\text{(iv)}\ \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad \text{(v)}\ \begin{pmatrix} 1 & 0 \\ 0 & -i/2 \end{pmatrix}$$

4. Show that any 2×2 unitary matrix can be written in the form

$$U = e^{i\phi}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. [Hint. Any such U can be written as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of four complex numbers. How must their absolute values be related to each other, and what conditions must their phases satisfy in order for U to be unitary?]

5. a) Show that the matrix

$$A = \begin{pmatrix} 1-3i & -4+2i \\ 4-2i & 4+3i \end{pmatrix}$$

is normal and that its eigenvalues are $\lambda_1 = 5 + 5i$ and $\lambda_2 = -5i$.

b) Find the normalized column eigenvectors $|x^1\rangle$ and $|x^2\rangle$.

c) Use $|x^1\rangle$ and $|x^2\rangle$ to construct a unitary matrix U which diagonalizes A:

$$A' = U^\dagger A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

d) Use $|x^1\rangle$ and $|x^2\rangle$ to construct the projectors $P_1 = |x^1\rangle\langle x^1|$ and $P_2 = |x^2\rangle\langle x^2|$ as Hermitian matrices. Check that $P_1^2 = P_1$, $P_2^2 = P_2$, $P_1P_2 = 0$, $P_1 + P_2 = I$, and $A = \lambda_1 P_1 + \lambda_2 P_2$, by using sums and products of these matrices.

6. Show that if an upper-triangular matrix, $A_{jk} = 0$ for $j > k$, is normal it is diagonal. [Hint. Start with the $j = 1, k = 1$ diagonal elements of $AA^\dagger$ and $A^\dagger A$. What is needed in order for them to be equal? Next consider $j = 2, k = 2$, and so forth.]

7. a) Show that if a and b are nonnegative real numbers, and P and Q are positive operators, then $aP + bQ$ is also a positive operator.

b) If P and Q are both positive, is the product PQ positive, either in general or in special cases? Discuss.

8. OPTIONAL. DO NOT TURN IN. Use the polar decomposition to prove that if A is any square matrix, then $B = A^\dagger A$ and $C = AA^\dagger$ have the same eigenvalues (including degeneracy), even though the eigenvectors are, in general different.

9. Find the unitary matrix U and the positive matrix R of the polar decomposition

$$A = \begin{pmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{pmatrix} = UR.$$

Show the details of your work; in particular, check that R is positive and U is unitary. You are welcome to enlist the aid of Mathematica or Maple (not really needed, but helpful for avoiding or correcting algebraic mistakes). In any case, write up your solution in legible form and explain your reasoning.

10. Suppose that $A = SDS^{-1}$ is an $N \times N$ matrix, S a similarity matrix, and D is diagonal with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_N$ on the diagonal. There is more than one way to define the logarithm of A. One can write

$$\ln A = S(\ln D)S^{-1}, \tag{1}$$

where $\ln D$ is the diagonal matrix with $\ln \lambda_1, \ln \lambda_2, \ldots$ on the diagonal. Or one can use a power series

$$\ln A = (A - I) - \frac{1}{2}(A - I)^2 + \frac{1}{3}(A - I)^3 - \cdots. \tag{2}$$

Find conditions on $\lambda_1, \lambda_2, \ldots$, assuming that they are complex numbers, such that the series (2) converges, and show that if it converges the two definitions give the same answer. (A series of matrices converges if each matrix element is given by a convergent series.) Will (1) still make sense for choices of the λ's for which (2) does not converge? [Hint. Start with simple cases, such as $N = 1$ or A diagonal, in order to get an idea of what is going on.]

11. OPTIONAL. DO NOT TURN IN. Riley, Hobson, Bence (2d ed), Exercise 8.34, slightly modified.

Solve the following simultaneous equations for x_1, x_2 and x_3 by using row reduction to form an upper-triangular matrix, and then solve the system starting with x_3 and working back to x_2 and x_1.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1, \\ 3x_1 + 4x_2 + 5x_3 &= 2, \\ x_1 + 3x_2 + 4x_3 &= 3. \end{aligned}$$

12. Riley, Hobson, Bence (2d ed), Exercise 8.36

Find the condition(s) on α such that the simultaneous equations

$$\begin{aligned} x_1 + \alpha x_2 \phantom{{}+ 3x_3} &= 1, \\ x_1 - x_2 + 3x_3 &= -1, \\ 2x_1 - 2x_2 + \alpha x_3 &= -1 \end{aligned}$$

have (a) exactly one solution, (b) no solutions, or (c) an infinite number of solutions; give all solutions where they exist.