33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 6.
Due Friday, Oct. 7

READING: (Needed for this assignment) Kreyszig Ch. 6, except Sec. 6.5. Linear Algebra Supplement, Secs. 1 through 10

READING AHEAD: (Needed for next assignment) Kreyszig Secs. 6.5, 7.1, 7.3, 7.4, 7.5. Linear Algebra Supplement, Secs. 11 through 17

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. DO NOT TURN IN. You should know how to do Nos. 1-10 in Kreyszig Sec. 6.8: identify whether some object is a vector space, and if so find the dimension and a basis.

3. DO NOT TURN IN. You should know how to do Nos. 13 to 21 in Kreyszig Sec. 6.8: norms and orthogonal complements. You can make No. 20 more interesting by replacing Kreyszig's real space with a complex space.

4. a) Find the norm of $\mathbf{v} = \begin{pmatrix} 1 \\ 1+i \\ -2i \end{pmatrix}$.

b) $\mathbf{v}$ spans a one-dimensional subspace S of C^3. Find a basis (it need not be orthonormal) for the orthogonal complement, $S^\perp$ of S.

c) $S^\perp$ is the null space of a certain transformation represented by a 2×3 matrix $\mathbf{A}$. Find a nontrivial $\mathbf{A}$ in the sense that neither row is zero. [Hint: What is the connection with $\mathbf{v}$?]

5. Let $\mathcal{P}_n$ be vector space of complex polynomials $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$ of degree n or less, and define $|e^j\rangle = z^j$ for $j = 0, 1, 2, \ldots n$ (with $z^0 = 1$) to be the *standard basis* of this $(n+1)$-dimensional space.

a) Why must $\mathcal{P}_n$ be defined using degree n *or less*; i.e., why do the polynomials of degree n not form a vector space?

b) Which of the following collections of polynomials are subspaces of $\mathcal{P}_n$? If the collection is a subspace, find its dimension and a basis for the subspace. If the collection is not a subspace, find its span (the smallest subspace that contains it).

(i) All $f(z)$ such that $f(1) = 0$.

(ii) All $f(z)$ such that $f'(0) = 1$

(iii) All $f(z)$ such that $f(z) = f(-z)$

c) Let $D := d/dz$ be the derivative operator: $Df(z) = f'(z)$. Find its range and its null space, and check that its rank plus its nullity is $n+1$.

d) Find the matrix $\mathbf{D}$ of D in the standard basis $|e^j\rangle$ defined above, using the formula $D|e^j\rangle = \sum_k D_{kj}|e^k\rangle$, and also its matrix $\hat{\mathbf{D}}$ in the basis $|f^k\rangle = z^k/k!$. Find the matrices $\mathbf{S}$ and $\mathbf{S}^{-1}$ for the similarity transformation $\hat{\mathbf{D}} = \mathbf{S}^{-1} \cdot \mathbf{D} \cdot \mathbf{S}$.

e) Show that for a given z_0, $L_0(f) = f(z_0)$ is a linear functional. (I.e., L_0 assigns to every $f(z) \in \mathcal{P}_n$ its value at $z = z_0$.) Next show that if z_0, z_1, $\ldots z_k$ are distinct complex numbers, the collection of $n+1$ linear functionals $\{L_k(f) = f(z_k)\}, k = 0, 1, 2, \ldots n$, is linearly independent. Hint. Can you construct a polynomial that vanishes at z_1, z_2, $\ldots z_n$, but not at z_0?

f) Define the linear operator $M : \mathcal{P}_n \to \mathcal{P}_n$ through $(Mf)(z) = f(z+c)$, where c is a complex number. What is the rank of M? Find its matrix $\mathbf{M}$ in the standard basis. [Hint. The pattern for the general case should emerge if you work out what happens for $n = 2$ and 3. Look for binomial coefficients.] What is the matrix $\mathbf{M}^{-1}$? [Hint: What is M^{-1}?]

g) One can define an inner product on $\mathcal{P}_n$ using

$$\langle g|f\rangle = \frac{1}{2\pi}\int_0^{2\pi} g^*(e^{i\theta})f(e^{i\theta})\,d\theta.$$

Show that this has the properties of an inner product; do not overlook the requirement that $\langle f|f\rangle = 0$ implies $f = 0$. What is $\langle g|f\rangle$ in terms of the coefficients of $|f\rangle$ and $|g\rangle$ in the standard basis?

h) Any linear functional on $\mathcal{P}_n$ can be written in the form $|f\rangle \to \langle g^0|f\rangle$. Find the polynomial $g^0(z)$ in the case that $L_0(f) = f(z_0)$. [Hint: Use the solution to (g).]

6. Let $|x\rangle$ and $|y\rangle$ be unit vectors along the x and y axis in the two-dimensional plane, with inner products

$$\langle x|x\rangle = 1 = \langle y|y\rangle, \quad \langle x|y\rangle = 0,$$

and let A be a rotation by $+45°$:

$$A|x\rangle = (|x\rangle + |y\rangle)/\sqrt{2}, \quad A|y\rangle = (-|x\rangle + |y\rangle)/\sqrt{2}.$$

a) Find the matrix $\mathbf{A} = (A_{ij})$ of A in the basis $|e^1\rangle = |x\rangle$, $|e^2\rangle = |y\rangle$ using the definition $A|e^j\rangle = \sum_i A_{ij}|e^i\rangle$. Check that $\mathbf{A}$ is real orthogonal by evaluating the matrix product $\mathbf{A}^{\mathrm{T}}\mathbf{A}$. Find $\mathrm{Tr}(\mathbf{A})$ and $\mathrm{Det}(\mathbf{A})$.

b) Find the matrix $\mathbf{A}'$ of A in the basis $|f^1\rangle = |x\rangle$, $|f^2\rangle = |x\rangle + |y\rangle$, using $A|f^j\rangle = \sum_i A'_{ij}|f^i\rangle$. Is $\mathbf{A}'$ a real orthogonal matrix? Find its trace and determinant.

c) Find the dual basis $|\bar{f}^j\rangle$ to $|f^j\rangle$ (in terms of $|x\rangle$ and $|y\rangle$), and check that

$$A'_{ij} = \langle \bar{f}^i|A|f^j\rangle.$$

d) Find the similarity matrix $\mathbf{S}$ such that $|f^j\rangle = \sum_i S_{ij}|e^i\rangle$, and its inverse $\mathbf{S}^{-1}$, and then check that $\mathbf{A}' = \mathbf{S}^{-1}\cdot\mathbf{A}\cdot\mathbf{S}$ by working out the matrix product.

7. For each of the following matrices find the rank r. Then find a set of r column vectors which span its range, and for (a), (c), and (d) a set of column vectors which form a basis of its null space. Give some indication of your method; do not just write down an answer.

$$\text{(a)}\ \begin{pmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{pmatrix} \quad \text{(b)}\ \begin{pmatrix} m & n & p \\ n & m & p \end{pmatrix} \text{ for } m^2 \neq n^2 \quad \text{(c)}\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{(d)}\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\text{(e)}\ \begin{pmatrix} 1 & -1+i & 2-i & -1+2i \\ 1 & -1 & 2 & -1 \\ -1 & 2 & -3 & 3 \end{pmatrix}$$