33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 5.
(Not to be turned in)

ANNOUNCEMENT: There will be an hour exam on Wednesday, Sept. 28 at 3:30. It will cover the part of the course devoted to complex variables and analytic functions: the assigned material in Chs. 12, 13, 14, and 15 of Kreyszig, along with supplementary material on branch points and branch cuts.

READING: Judging from your response in class, everyone should be familiar with the material on linear algebra in Kreyszig Secs. 6.1 through 6.6. You should glance through these to refresh your memory, and look at some problems to make sure you know how to do them. Some of the same material will be discussed in lectures from an abstract point of view, for which see the *Linear Algebra Supplement* (2005) by Griffiths. As the last is not entirely ready, it will be coming out in different versions.

READING AHEAD: Kreyszig Ch. 7. Again, a lot of this material should be familiar, but reviewing it is a good idea. (Sec. 7.2 is not part of the course, but is worth looking at in terms of physical applications that give rise to eigenvalue problems.) The abstract approach to eigenfunctions and eigenvalues, along with material going a bit beyond that in Ch. 7 of Kresyzig, will be in the *Linear Algebra Supplement*.

EXERCISES: *DO NOT TURN IN!*

In reviewing for the hour exam, you may find it helpful to look at the exercises at the ends of chapters 12 through 15, labeled "Chapter Review", in Kreyszig.

Fall 2003 hour exam:

1. Consider the problem of finding the principal value of the integral

$$\fint_{-\infty}^{\infty} \frac{dx}{(1+x^2)\cos x}.$$

a) Briefly define what is meant by the (Cauchy) principal value of an integral. The principal value only exists for certain types of integrand; in particular it would not exist if $\cos x$ were replaced with $(\cos x)^2$ in the example above. Discuss.

b) In order to evaluate an integral of this type using the method of residues it is important that the integrand satisfy some condition(s) on a large semicircle in the upper half plane. Discuss this for the case of the simpler integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

(A similar argument can be constructed with $\cos x$ present in the denominator, but it is more subtle, and you are not asked to discuss that case.)

c) Evaluate the residue of

$$f(z) = \frac{1}{(1+z^2)\cos z}$$

at all poles lying off the real axis and *just one* pole on the real axis.

d) Use your results in (c) to evaluate the principal value of the integral specified at the beginning of this problem. Explain why you do not need to evaluate residues at other poles on the real axis. (You can give a plausible argument; you do not need to provide a proof.) Express the integral in terms of the hyperbolic cosine of some quantity.

2. a) Where are the singular points of the function

$$f(z) = \frac{\sqrt{1+z^2}}{z^3},$$

and what is the type (pole, branch point, etc.) of each one? In the case of a pole, specify its order.

b) Expand $f(z)$ as a Laurent series $\sum a_n z^n$ about $z = 0$, assuming the branch in which $f(z) \approx z^{-3}$ when z is small, and find the explicit value of a_n for all $n \leq 2$. What is the region in the complex plane where the Laurent series converges? Explain.

c) Extend the analytic function defined by the Laurent series in (b) to the entire complex plane by making two branch cuts extending to $+i\,\infty$ and $-i\,\infty$ lying outside the region where the Laurent series converges. Sketch the z plane with these branch cuts.

Now suppose one starts from near $z = 0$ with the function defined by the Laurent series in (b), and continues it analytically along a path which circles the upper branch point in a counterclockwise direction and returns to the vicinity of the origin. What is the relationship, in the neighborhood of $z = 0$, between the new function of z obtained as the result of this analytic continuation, and the function one started out with? In particular, how are the Laurent series for the two functions near $z = 0$ related to each other? Be explicit, and give reasons for your answer, though you do not have to provide a proof.

d) Consider a line $z = it + \epsilon$, $0 < t < \infty$ in the complex plane, where $\epsilon > 0$ can be considered negligibly small; it simply ensures the path is to the right of the branch cut. Sketch the real and imaginary parts of $f(z)$, using the same branch as in (b) extended to an analytic function in the cut plane as in (c), along this path as a function of t. The sketches do not have to be fancy, but should indicate that you know what you are doing. If the real and imaginary parts are on the same sketch, indicate which is which. [Hint. Suppose you start at $z = R$, $R \gg 1$, on the real axis, and move counterclockwise on a circle of radius R centered at the origin till you reach the imaginary axis. What happens to $f(z)$?]