33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 4.
Due Friday, Sept. 23

ANNOUNCEMENT: There will be an hour exam on Wednesday, Sept. 28 at 3:30. It will cover the part of the course devoted to complex variables and analytic functions: the assigned material in Chs. 12, 13, 14, and 15 of Kreyszig, along with supplementary material on branch points and branch cuts.

READING: Kreyszig Secs. 13.2, 13.3, 13.4; 15.3, 15.4

READING AHEAD: Kreyszig Ch. 6

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. OPTIONAL. a) Sketch the following contour C in the complex z plane, with arrows to indicate the sense. It consists of two pieces: C_1 is a semicircle of radius 2 centered at $z = 1$, extending from $z = 1 - 2i$ to $z = 1 + 2i$, and passing through $z = 3$; C_2 is a straight line from $z = 1 + 2i$ to $z = 1 - 2i$ passing through $z = 1$.

b) Parametrize the contour (use separate parametrizations for C_1 and C_2) using t as the parameter, and evaluate the integral

$$\oint_C \bar{z}\, dz$$

in the counterclockwise sense around the closed contour.

c) Use the same parametrization as in (b) to evaluate

$$\oint_C z\, dz,$$

and show that it has the value you expect.

3. a) Find the maximum absolute value of $f(z) = z^2 e^{iz}$ on the semicircle C defined by

$$z = 2e^{i\theta}, \quad -\pi/2 \leq \theta \leq \pi/2,$$

and use this to find an "ML" bound (Kreyszig, p. 711) on the absolute value of the integral

$$I = \int_C f(z)\, dz.$$

b) Evaluate the integral in closed form, assuming a counterclockwise orientation of C (from $-2i$ to $+2i$), and check that the result satisfies the bound. [Hint: What is the indefinite integral of $f(z)$?]

4. OPTIONAL. (a) Evaluate each of the following complex integrals, where S is the counterclockwise semicircle $|z| = R$ in the upper half plane beginning at $z = +R$, passing through $z = iR$, and ending at $z = -R$.

$$\text{(i)} \int_S \bar{z}\, dz, \quad \text{(ii)} \int_S |z|\, dz, \quad \text{(iii)} \int_S z\, dz, \quad \text{(iv)} \int_S z^{-1}\, dz.$$

(b) Cauchy's integral theorem implies that some of the integrals in (a) will have the same value if S is replaced by a different path S' with the same end points. Which of the integrals will be the same, and what additional restrictions, if any, must S' satisfy? Why does Cauchy's theorem not apply to the other integrals?

5. The function

$$f(z) = \frac{\cos \pi z}{z^2 - 1}$$

can be developed in a Laurent series about the center $z = 1$ for $0 < |z-1| < R$, and also for $R < |z-1| < \infty$.

a) What is R? Explain.

b) Find the first three non-zero terms in the Laurent series for $0 < |z-1| < R$. Express your answer in terms of $w = z - 1$. Remember that $\cos(\pi + z) = -\cos z$.

c) Evaluate

$$\oint_C f(z)\, dz$$

counterclockwise around the two contours

$$\text{(i) } |z-1| = 1/2, \quad \text{(ii) } |z-1| = 5.$$

6. Simple residues. You should be able to do any of Kreyszig Sec. 15.3, Nos. 1-10 with the book closed; in particular, you should know how to derive his formula (5) should the need arise. Turn in solutions to No. 2 and No. 4.

7. Residue integration of simple functions. Try some of those in Kreyszig Sec. 15.3, Nos. 12 to 20 to be sure you know how to do them. Turn in the solution No. 14.

8. Riley, Hobson and Bence, 2d edition, Exercise 20.17.

By considering the real part of

$$\int_C \frac{-iz^{n-1}\, dz}{1 - a(z + z^{-1}) + a^2},$$

where $z = \exp i\theta$, n is a non-negative integer, and C a contour that you should specify, evaluate

$$\int_0^\pi \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\, d\theta,$$

for a real and > 1. [Hints: Pole at $z = 1/a$. Integral is $\pi a^{-n}(a^2-1)^{-1}$.]

9. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{-i2x}}{x^2+x+1}\,dx$$

by the method of residues, by closing the contour in the *lower* half plane (mirror image in the x axis of Kreyszig's Fig. 360). Indicate briefly why given this integrand one must close the contour in the lower rather than in the upper half plane. Pay attention to signs. Show your work.

10. Principal value integral using residues: Kreyszig Sec. 15.4, No. 24.

11. Riley, Hobson and Bence, 2d edition, Exercise 20.25.

Using a cut plane with a cut along the positive real axis, and a contour of the sort shown in Fig. 20.21 of Riley, Hobson and Bence, prove that if α is real and $0 < \alpha < 1$, then

$$\int_0^{\infty} \frac{x^{-\alpha}}{1+x}\,dx = \frac{\pi}{\sin \pi\alpha}.$$

Note that the function $z^{-\alpha}$ is multivalued, so you need to be quite specific in indicating which branch you are using; it helps to state explicitly what its values are on the top and bottom of the cut along the positive real axis. What role is played by the fact that α lies inside the limits $0 < \alpha < 1$?