33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 3.
Due Friday, Sept. 16

ANNOUNCEMENT: There will be an hour exam on Wednesday, Sept. 28 at 3:30. It will cover the part of the course devoted to complex variables and analytic functions: the assigned material in Chs. 12, 13, 14, and 15 of Kreyszig, along with supplementary material on branch points.

READING: Kreyszig Secs. 12.8, 12.10; 13.1; 15.1 15.2
15.1 through 15.4; material on branch points/cuts from Churchill.

READING AHEAD: Kreyszig Secs. 13.2, 13.3, 13.4; 15.3, 15.4

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. Kreyszig Sec. 15.2, Nos. 2, 4, 10, 14.

3. a) Find the singularities of $f(z) = \sin(\sqrt{1-z^2})$ in the complex z plane. If there are one or more branch points, indicate a suitable branch cut or cuts to make the function single valued.

b) Same question for $g(z) = \cos(\sqrt{1-z^2})$.

4. OPTIONAL. DO NOT TURN IN. a) Find explicit expressions for the coefficients in the power series expansion of $\operatorname{Ln} z$ about the point $z_0 = -1 + i$, and check that the radius of convergence is what you expect it to be.

b) Does this power series converge to $\operatorname{Ln} z$ for all z inside the disk of convergence $|z - z_0| < R$, and if not, what does it converge to? [Hint: You may find it useful to sketch the relevant part of the complex plane.]

5. Kreyszig Sec. 15.2 No. 8. DO NOT TURN IN. If you can do this exercise with minimal reference to the text you have understood some very important ideas presented in this section.

(continues on the other side)

6. Consider the multiple-valued function

$$w = f(z) = (z+1)^{1/3}.$$

a) To make it single-valued, use a branch cut extending to $-\infty$ along the negative real axis, and employ the branch for which w is real when $z = 2$. Let us call this single-valued function, which is analytic everywhere except on the cut, $f_a(z)$. Discuss what values $w = f_a(z)$ takes when z is on a unit circle $|z+1| = 1$ centered on the branch point, by using a sketch in the z plane and a corresponding sketch in the w plane, with a few points labeled so the correspondence is clear. Find the discontinuity Δf_a, the limit of $f_a(z+i\epsilon) - f_a(z-i\epsilon)$ as ϵ goes to zero through positive values along the cut, as a function of $|z+1|$.

b) Repeat (a), again using the branch of the function which is real when $z = 2$, but now with a branch cut going to $+i\infty$ parallel to the positive imaginary axis. Let us call this function $f_b(z)$. As well as making the sketch, find Δf_b, the value of $f_b(z)$ just to the right of the branch cut minus its value just to the left. For what values of z is it the case that $f_b(z) = f_a(z)$? For what values are they different?

c) Repeat (a) with the same branch cut extending to $-\infty$, but now use the branch of w which has a positive imaginary part when $z = 2$, and call this the function $f_c(z)$. For what values of z is $f_c(z)$ equal to or different from $f_a(z)$? Suppose you were to analytically (smoothly!) continue $f_c(z)$ across the branch cut starting from the top side. How far would you need to go before you arrived back on the branch used in (a), i.e., $f_a(z)$?

7. Kreyszig Sec. 15.1, Nos. 2 and 4.

8. Kreyszig Sec. 15.1 No. 10. In doing an expansion about $z = z_0$ with $z_0 \neq 0$, it is often easier to introduce a new variable $w = z - z_0$—in this case $w = z - 1$—and work out the expansion on powers of w, i.e., working out the coefficients c_n in the sum $\sum_{-\infty}^{\infty} c_n w^n$. Expressing the general term in closed form is a bit of a mess; it will suffice if you work out c_n for $n \leq 3$ in a manner which shows that if you needed to obtain additional terms you would be able to find them.

9. Check that you know now to do Nos. 5 to 9 in Kresyzig Sec. 12.8. DO NOT TURN IN.

10. Check that you know how to solve exercises 10-14 and 20-29 in Kresyzig Sec. 12.8. Turn in solutions to Nos. 13 and 26.