33-759 Introduction to Mathematical Physics
Fall Semester, 2005
Assignment No. 1.
Due Friday, Sept. 2

READING: Kreyszig Secs. 12.1 through 12.5

READING AHEAD: Kreyszig Secs. 12.6, 12.7; 14.1 through 14.4

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course. You will find a sample at the end of the problem set.

2. You should know how to do the exercises at the end of Kreyszig's Sec. 12.1 and Sec. 12.2 without looking up formulas in the book. Try one or two at random; note that solutions to odd-numbered problems are given in the back of the book. Do not turn them in.

3. (Kreyszig 12.1, No. 8 modified.) Let $z_1 = 2 + 3i$, $z_2 = -3 - 5i$. Find $(z_1 - z_2)/(z_1 + z_2)$ by two methods: (i) Multiply the numerator and denominator by the complex conjugate of the latter. (ii) Convert both numerator and denominator to polar form (e.g., numerically on a pocket calculator), and carry out the division using the polar forms. The results should agree.

4. Kreyszig 12.2, No. 22: Find and plot all roots $\sqrt[3]{8i}$.

5. The assigned problem was Kreyszig 12.1, No. 20: Prove that if $zw = 0$ for complex numbers z and w, then $z = 0$ or $w = 0$. Here is Willy's defective proof. Find out where it is wrong, and then fix it.

P1. Assume $zw = (x + iy)(u + iv) = 0 = (xu - yv) + i(yu + xv)$.

P2. A complex number is zero only if its real and imaginary parts vanish, so $xu = yv,\ yu = -xv$.

P3. Divide the first equation by x, the second by y, assuming $x \neq 0$, $y \neq 0$, to get $u = (y/x)v = -(x/y)v$.

P4. If $v \neq 0$, this means $y/x = -x/y$, or $y^2 = -x^2$, which is obviously impossible if $x \neq 0$, $y \neq 0$.

P5. Thus if $z \neq 0$, we see that $v = 0$, but as $u = (x/y)v$, this implies that $u = 0$, and thus $w = u + iv = 0$.

P6. The symmetrical argument shows that if $w \neq 0$, $z = 0$. QED

6. OPTIONAL. DO NOT TURN IN.

a) On the qualifying examination Willy wrote

$$|a + ib|^2 = |a|^2 + |b|^2.$$

The grader marked it as wrong. Why? Under what conditions (please state all of them) on the complex numbers a and b is this formula correct?

b) On the same examination Cindy lost points because she wrote z^2 instead of $|z|^2$, and the grader was not sympathetic. Show that

$$\alpha(z) = z^2/|z|^2$$

is in general not equal to 1. What are its possible values?

7. The usual requirement for the domain of definition of an analytic function is that it be a connected open set of the complex plane. For each of the following sets, make a sketch of the set, state whether it is open and connected, and if it is not open and connected, suggest a small change — add some points or take away some points — which will make it an acceptable domain. Sets are denoted here by $\{\ldots\}$, and $A\backslash B$ means the set of points that are in A but *not* in B.

(a) $\{x > 0 \text{ and } |z| < 1\}$, (b) $\{x \geq -1/2 \text{ and } |z| < 1\}$, (c) $\{y = 0 \text{ and } 0 < x < 1\}$,
(d) $\{|z| < 1\} \cup \{|z-2| < 1\}$, (e) $\{|z| < 1\}\backslash\{z = 1/2\}$,
(f) $\{|z| < 1\}\backslash\{z = 1/2, 1/3, 1/4, \ldots\}$.

8. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Write the derivative $f'(z)$ in terms of derivatives of $u(x, y)$ alone. Also write it in terms of derivatives of $v(x, y)$ alone.

9. Find the analytic function $f(z)$ whose imaginary part is $(y \cos y + x \sin y)e^x$, with $z = x + iy$. Indicate the nature of any undetermined constants.

10. OPTIONAL. DO NOT TURN IN.

If $w = f(z)$ is analytic, under what conditions will its inverse function $z = g(w)$ be analytic? How will the derivative $g'(w)$ of the inverse function be related to the derivative of $f(z)$?

11. a) Sketch in the z plane the region

$$0 \leq |z| \leq 1.5, \quad \pi/4 \leq \operatorname{Arg} z \leq 3\pi/4.$$

b) Find and sketch the image of this region in the w plane under the map $w = z^2$.

c) Is this map conformal at the points $z_0 = 0$, $z_1 = 1.5e^{i\pi/4}$, and $z_2 = 1.5e^{i3\pi/4}$? Discuss in terms of the curves bounding the original region and its image, as shown in your sketches, and the conditions under which an analytic function yields a conformal map.

12. Kreyszig 12.5 No. 12. Find all points at which the mapping $w = (z^3 - a)^2$ is not conformal. Assume a is some complex number (not necessarily real).

13. Kreyszig 12.5 No. 16. Find a parametric representation of the curve $x^2 + 9y^2 = 9$

Sample answer to Exercise 1 by Willy Smart.

I glanced at Sec. 12.1. Kid stuff. I learned 12.2 in high school. Well, to be honest, we didn't do all that root business, and I reviewed example 4, even though I'm sure we did it in advanced calculus. The definition of analytic functions in 12.3 is something I'd seen before, but review doesn't hurt. Same with Cauchy-Riemann in 12.4.

However, Sec. 12.5 was not so easy. I tried to work through example 1, and got thoroughly confused. Where do those curves in Fig. 307 come from? Can one somehow get Maple to plot them? Does one actually have to sketch curves, as in Fig. 308, in order to find the angle between the tangents, or is there some way to do that analytically? And Fig. 312 has just *got* to be wrong — surely you can't get something with a point on it out of a smooth curve.

Complaints: Instructor tells too many jokes. Also, the pace of the course has been much too *slow*. Except for conformal mapping, which was a bit too fast.