

# Notes on Quantum Information Theory

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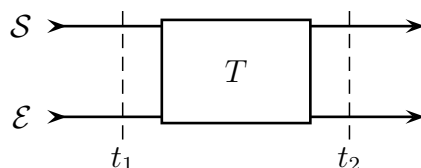
References:

*Quantum Computation and Quantum Information* by Nielsen and Chuang (Cambridge, 2000), Secs. 2.2.3 to 2.2.6, 2.2.8, 8.2.3

## 7 Measurements: General Interaction Model

### ■ General interaction model

★ The following *general interaction model* occurs in a number of different contexts in quantum information theory, so it is worthwhile noting some of its features. A *system*  $\mathcal{S}$  and *environment* or *apparatus*  $\mathcal{E}$  interact during the time interval between  $t_1$  and  $t_2$ , and the time development of the combination, viewed as isolated from the rest of the world, is given by a unitary time development operator  $T = T(t_2, t_1)$  acting on the Hilbert space  $\mathcal{H}_s \otimes \mathcal{H}_e$ .



The initial state is

$$|\Psi_1\rangle = |\psi\rangle \otimes |e\rangle, \quad (7.1)$$

i.e., the system is in the state  $|\psi\rangle$  and the environment in the state  $|e\rangle$ .

◦ If the environment is described at  $t_1$  by a density operator, one can “purify” it by introducing a fictional reference system, which is then thought of as part of the environment  $\mathcal{E}$ .

★ The general interaction model is a model system for:

- A *measurement*. Here  $\mathcal{E}$  is the apparatus with a *pointer basis*  $\{|e_m\rangle\}$  at  $t_2$ , and one is interested in the correlation between the initial system state  $|\psi\rangle$  and the final pointer state  $|e_m\rangle$ , the *outcome* of the measurement.

- A *preparation*. One is interested in the correlation between the final pointer state  $|e_m\rangle$  and the state of the system  $\mathcal{S}$  at  $t_2$ , usually with a view of describing what happens to  $\mathcal{S}$  at a later time, when it may interact with something else.

- *A quantum channel.* One is interested in the correlation between the system state at  $t_2$  and the initial  $|\psi\rangle$ , without reference to, or simply ignoring the final state of the environment.

- *Decoherence.* Same as the quantum channel, but one is typically interested in a specific initial  $|\psi\rangle$ , whereas for a quantum channel  $|\psi\rangle$  is regarded as variable.

### ■ Histories

★ In order to turn the general interaction model into a stochastic model in which one can assign probabilities, it is necessary to specify a family of histories with initial state (7.1). The choice of later events depends upon what one is interested in. When discussing measurement and preparation, one includes the  $\{|e_m\rangle\}$  (pointer) basis at  $t_2$ . For measurements, one typically specifies some basis  $\{|s_k\rangle\}$  of  $\mathcal{H}_s$  at  $t_1$ , and for preparations a basis of  $\mathcal{H}_s$  at  $t_2$ , or else the state of  $\mathcal{S}$  conditional on a particular  $|e_m\rangle$ , see (7.2).

◦ In the case of measurements, even if  $|\psi\rangle$  is incompatible with  $\{|s_k\rangle\}$ , one can still introduce histories in which  $|\psi\rangle$  refers to the state of  $\mathcal{S}$  at a slightly earlier time. It is then necessary to check the consistency of the histories, because more than two times are involved.

### ■ Kraus operators

★ Kraus operators are a mathematical tool that is often useful for analyzing a general interaction model. The idea is to expand

$$|\Psi_2\rangle = T|\Psi_1\rangle = \sum_m |\bar{\phi}_m\rangle \otimes |e_m\rangle \quad (7.2)$$

in the orthonormal basis  $\{|e_m\rangle\}$  of  $\mathcal{H}_e$ , with coefficient kets  $\{|\bar{\phi}_m\rangle\}$  in  $\mathcal{H}_s$ , which are (in general) neither normalized nor orthogonal. They can be written in the form

$$|\bar{\phi}_m\rangle = K_m|\psi\rangle, \quad (7.3)$$

where the  $K_m$  are *Kraus operators* (also called “measurement operators” in Sec. 2.2.3 of QCQI, but simply “operators” in Sec. 8.2). They are linear operators that depend on  $T$ ,  $|e\rangle$ , and the basis  $\{|e_m\rangle\}$ , but not on  $|\psi\rangle$ .

◦ Note that Kraus operators map  $\mathcal{H}_s$  to itself, thus they are operators on  $\mathcal{H}_s$ , even though their actual form is determined by states on  $\mathcal{H}_e$ . They need not be unitary or Hermitian; the only constraint is the sum condition (7.5).

- The probability of a particular outcome  $|e_m\rangle$  at  $t_2$  is

$$\Pr(e_m) = \langle \Psi_2 | [e_m] | \Psi_2 \rangle = \langle \bar{\phi}_m | \bar{\phi}_m \rangle = \langle \psi | K_m^\dagger K_m | \psi \rangle. \quad (7.4)$$

The normalization condition

$$\sum_m K_m^\dagger K_m = I, \quad (7.5)$$

where  $I$  is the identity on  $\mathcal{H}_s$ , is easily derived using the unitarity of  $T$ , and ensures that the probabilities in (7.4) sum to 1.

- The *correlation* between the measurement outcomes (or environmental states) and the system  $\mathcal{S}$  at  $t_2$  can be expressed by saying that the state of the latter *conditional upon*  $|e_m\rangle$  is  $|\bar{\phi}_m\rangle$ , or, if one normalizes it,

$$|\phi_m\rangle = |\bar{\phi}_m\rangle / \sqrt{\langle \bar{\phi}_m | \bar{\phi}_m \rangle} = K_m |\psi\rangle / \sqrt{\text{Pr}(e_m)}. \quad (7.6)$$

It is usually best to think of such a conditional state as a pre-probability.

#### ■ POVM

- ★ A collection  $\{Q_m\}$  of positive operators which sum to the identity,

$$\sum_m Q_m = I, \quad (7.7)$$

is called a *positive operator-valued measure*, or POVM.

- The fancy name makes a certain amount of sense when one is dealing with infinite-dimensional Hilbert spaces.

- From the fact that  $A^\dagger A$  is positive for any operator  $A$ , and from the normalization condition (7.5), it follows that the operators

$$Q_m = K_m^\dagger K_m. \quad (7.8)$$

obtained from Kraus operators form a POVM.

- ★ One can regard a POVM as a tool for calculating probabilities of measurement outcomes in terms of an initial state; thus (7.4) can be written as

$$\text{Pr}(e_m) = \langle \psi | Q_m | \psi \rangle. \quad (7.9)$$

- It is best to regard a POVM as a pre-probability whose relation to physical reality is similar to that of a density operator. Only in special cases can one associate a particular  $Q_m$  with some quantity that has been measured.

- Note that the POVM (7.8) is of no use for discussing the state of  $\mathcal{S}$  at  $t_2$ , so it cannot be used in describing preparations.

#### ■ Quantum channel

- ★ In the case of a quantum channel, one is interested in the correlation between the initial state  $|\psi\rangle$  and the final state of  $\mathcal{S}$  at time  $t_2$ , without reference to the state of the environment. This usually requires the use of a density operator to describe the state at  $t_2$ . (More about this subject later.)