33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Sample Final Questions — only the post-midterm-2-pieces

1. Gas particle(s) in a trap

A particle of mass m is held in a three-dimensional spherically symmetric trap of potential energy $U(\boldsymbol{r}) = C|\boldsymbol{r}|$.

1. If the particle is at a temperature T, calculate its canonical partition function $Z(T, N = 1) \equiv Z(T, 1)$.

   Note: as usual, the abbreviation $\lambda_{th} := h/\sqrt{2\pi m k_{\rm B} T}$ cleans the final expression a little bit up.

2. Calculate the particle's free energy $F(T, N = 1) \equiv F(T, 1)$.

3. If we put N such particles into the trap, *which do not interact with one another*, what is now the canonical partition function $Z(T, N)$? Specifically, how does it relate to $Z(T, 1)$?

4. Calculate the associated free energy $F(T, N)$ in the limit $N \gg 1$ and express it formally using $F(T, 1)$.

2. Gas centrifugation

Let's learn how to enrich uranium. Let's also put this in writing here and put the pdf online, so that the NSA reads with us, and the folks at Fort Meade, Maryland, learn a bit of physics in the process. Can't hurt.

An ideal gas of molecules with mass m is confined in a cylinder of radius R and height h, which also rotates with an angular frequency ω around its symmetry axis. As a consequence, gas molecules a distance r away from the axis feel the centrifugal potential energy $U(r) = -\frac{1}{2}m\omega^2 r^2$ (in the rest frame of the cylinder, which we will use in all what follows).

1. What is the (normalized!) probability density $p(\boldsymbol{r})$ of finding a particular gas molecule at *position* $\boldsymbol{r}$ inside the cylinder, assuming that inside the cylinder we have the constant temperature T?

   Attention: $\boldsymbol{r}$ is a vector!

2. What is the (normalized!) probability density $p_m(r)$ of finding a particular gas molecule at *distance* r away from the axis?

   Attention: r is a scalar! Hint: transformation theorem!

3. What is the probability $p_{m,\varepsilon}$ of finding a particular gas molecule within a cylindrical region of radius ε from the axis? Expand this probability to lowest nontrivial order in ω.

   Hint: You will have to expand some exponentials; think carefully how far you have to push this expansion!

4. Let the cylinder now contain a mixture of two ideal gases: n light molecules of mass m and N heavy molecules of mass $M > m$. The average ratio of light versus heavy molecules is of course n/N, but on the cylinder axis (meaning, within the central region of radius ε) this ratio is enriched by some factor $\alpha > 1$ due to the centrifugation. To lowest nontrivial order in ω, and in the limit $\varepsilon \to 0$, calculate that enrichment factor and show that its deviation from unity, $\alpha - 1$, is proportional to the mass difference, $M - m$, and inversely proportional to the temperature, T.

   In real life, the gas we're talking about is uraniumhexafluoride, and the heavy gas molecules are ${}^{238}UF_6$, while the light ones are ${}^{235}UF_6$. You want the latter, because ${}^{235}U$ is the fissile isotope of uranium, but it's chemically identical to the non-fissile one, ${}^{238}U$. Moreover, ${}^{235}U$ occurs in natural uranium only with an abundance of 0.75%, much too little to sustain a chain reaction. For a typical nuclear reactor we need 3%-5%. Bomb-grade enrichment is somewhat arbitrarily defined as 90%—arbitrarily, because even at 20% ("highly enriched uranium") one can build an inefficient bomb. Whoops, we also said "bomb" in the same pdf file. Well, at least we didn't say "terrorism".

3. Polymer chain length fluctuations

Following up on homework problem 48: if we define $\chi_T := \frac{1}{\langle L\rangle}\left(\frac{\partial\langle L\rangle}{\partial\tau}\right)_T$ as the susceptibility of the chain length under tension (you might call it the polymer's "isothermal extensibility"), show how it relates to the fluctuations σ_L^2 of the chain's length!