Characteristic function of the Gaussian probability density

The probability density of a Gaussian (or "normal distribution") with mean μ and variance σ^2 is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\ . \tag{1}$$

Its *characteristic function* is defined to be the Fourier transform

$$\tilde{p}(k) = \left\langle \mathrm{e}^{\mathrm{i}kx}\right\rangle = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\mathrm{d}x\ \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{e}^{\mathrm{i}kx}\ . \tag{2}$$

To work this out, all we really need to do is to complete the square in the exponent of the integral:

$$\begin{aligned}
-\frac{(x-\mu)^2}{2\sigma^2} + \mathrm{i}kx &= -\frac{1}{2\sigma^2}\Big\{(x-\mu)^2 - 2\mathrm{i}k\sigma^2 x\Big\} \\
&= -\frac{1}{2\sigma^2}\Big\{x^2 - 2x\mu + \mu^2 - 2\mathrm{i}k\sigma^2 x\Big\} \\
&= -\frac{1}{2\sigma^2}\Big\{x^2 - 2x(\mu+\mathrm{i}k\sigma^2) + \mu^2\Big\} \\
&= -\frac{1}{2\sigma^2}\Big\{\underline{x^2 - 2x(\mu+\mathrm{i}k\sigma^2) + (\mu+\mathrm{i}k\sigma^2)^2} - (\mu+\mathrm{i}k\sigma^2)^2 + \mu^2\Big\} \\
&= -\frac{1}{2\sigma^2}\Big\{\Big[x-(\mu+\mathrm{i}k\sigma^2)\Big]^2 - \mu^2 - 2\mathrm{i}k\mu\sigma^2 + k^2\sigma^4 + \mu^2\Big\} \\
&= -\frac{1}{2\sigma^2}\Big\{\Big[x-(\mu+\mathrm{i}k\sigma^2)\Big]^2 - 2\mathrm{i}k\mu\sigma^2 + k^2\sigma^4\Big\} \\
&= -\frac{1}{2\sigma^2}\Big[x-(\mu+\mathrm{i}k\sigma^2)\Big]^2 - \frac{1}{2}k^2\sigma^2 + \mathrm{i}k\mu\ .
\end{aligned} \tag{3}$$

From this we immediately get

$$\begin{aligned}
\tilde{p}(k) &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\mathrm{d}x\ \mathrm{e}^{-\frac{1}{2\sigma^2}\left[x-(\mu+\mathrm{i}k\sigma^2)\right]^2 - \frac{1}{2}k^2\sigma^2 + \mathrm{i}k\mu} \\
&= \mathrm{e}^{-\frac{1}{2}k^2\sigma^2+\mathrm{i}k\mu}\underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\mathrm{d}x\ \mathrm{e}^{-\frac{1}{2\sigma^2}\left[x-(\mu+\mathrm{i}k\sigma^2)\right]^2}}_{=1} \\
&= \mathrm{e}^{-\frac{1}{2}k^2\sigma^2+\mathrm{i}k\mu}\ ,
\end{aligned} \tag{4}$$

where in the last step we exploited the standard integral over a shifted Gaussian, using the fact that this holds even if the shift [here: $-(\mu+\mathrm{i}k\sigma^2)$] is a complex number.