33-331 Physical Mechanics I. Fall Semester, 2008
Final Examination 12 December 2008.

NAME________________________________

INSTRUCTIONS. This examination consists of five problems, each of which has three parts and is worth a total of 60 points, for a grand total of 300 points. It is important to give a *brief* indication of your reasoning in all cases. Answers written down without explanation will receive little or no credit. Instead, answer in a way which shows that you *understand* the subject. If you do not understand what a question means, or think additional formulas would be helpful, please ask the instructor!

If you need more space, use the back side of the page, but indicate on the front side that the answer continues on the back side.

FORMULAS

$$\mathbf{r} = (r,\theta); \quad v_r = \dot r; \; v_\theta = r\dot\theta$$

$$\int_{x_1}^{x_2} f(y,y',x)\,dx : \quad \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}; \quad \frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot q_j}; \quad \frac{\partial H}{\partial q_j} = -\dot p_j, \quad \frac{\partial H}{\partial p_j} = \dot q_j$$

$$\mathbf{L} = \mathbf{R}\times\mathbf{P}; \quad \mathbf{N} = \mathbf{R}\times\mathbf{F}; \quad \dot{\mathbf{L}} = \mathbf{N}$$

$$V_{\text{eff}}(r) = U(r) + l^2/2\mu r^2$$

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1. Two masses m_1 and m_2 are placed on a smooth, frictionless table and connected with a spring in such a way that the potential energy is

$$U(r) = \tfrac{1}{2}k(r-r_0)^2,$$

where r is the distance between them (assume they are point particles), and k and r_0 are positive (nonzero) constants.

a) Write down the Lagrangian L in Cartesian coordinates, in terms of of x_1, y_1, x_2, y_2, and their time derivatives. Use this to derive an expression for L as a function of the position of the center of mass and the difference of positions,

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}; \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1,$$

and their time derivatives, using polar coordinates r, θ for $\mathbf{r}$. You may use the fact that

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

b) For the Lagrangian written in terms of $\mathbf{R} = (X, Y)$ and $\mathbf{r} = (r, \theta)$ find expressions for the generalized momenta and identify which of these are conserved quantities, explaining why they are conserved. Is there any other dynamical quantity that is conserved?

c) What are the conditions for circular orbits, i.e., each mass is moving in a circle? Consider the possibilities $r < r_0$, $r = r_0$, and $r > r_0$. Are circular orbits stable? Assuming $m_1/m_2 = 1/2$, and with the center of mass at the origin, make a sketch of the circles described by $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and $\mathbf{r}(t)$ in the case of a particular circular orbit, indicating where all three are at some particular time (e.g., by dots on their respective circles).

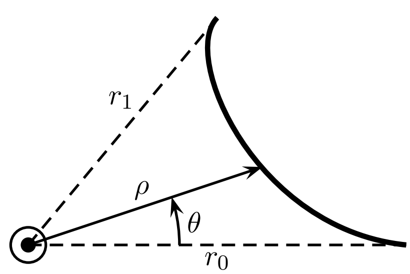
2. Angry Thartians, who inhabit an inner planet of the star Thartus, are about to launch an attempt to capture a recently discovered spacecraft, which they suspect comes from the distant star Ubrus and is being used to spy on their civilization. The alien spacecraft is moving with speed v_0 in a circular orbit a distance R from the center of Thartus, mass M_T. So the time has come for the Ubrians on the spacecraft to fire an emergency rocket to produce a Δv sufficient to escape from Thartus' gravity.

a) What is the minimum Δv required to (i) achieve a parabolic orbit or (ii) a hyperbolic orbit with a speed equal to v_0 when the space station is very far away from Thartus? Assume that the Δv is produced by firing the rocket for a time very much shorter than the period of the circular orbit. Indicate the direction of the Δv as well as its magnitude. Express your answers in terms of v_0 and possibly other quantities.

b) The commander would prefer, as it means a quicker return to Ubrus, to fire the rocket in a direction perpendicular to what you found for (a). Recalculate the Δv for cases (i) and (ii).

c) The space station has a mass of m_S kg, and the emergency rocket a mass of $m_R + m_F$, where m_F is the mass of the fuel. Assume the rate at which the fuel is burned up is α kg/s, and the exhaust speed relative to the rocket is u m/s. What must m_F be in terms of other quantities so that the if all the fuel is burned the result will be a Δv equal to $3u$? Work out the answer starting with the basic differential equation for rocket motion in free space, which is $m\,dv + u\,dm = 0$.

3. Consider the brachistochrone problem for an object moving not in a uniform gravitational field but in one due to a spherical mass located at the origin in polar coordinates, giving rise to a gravitational potential $\Phi(r) = -k/r$. Given two points, r_0, θ_0 with $\theta_0 = 0$, and r_1, θ_1 with $\theta_1 > 0$, see sketch, the brachistochrone problem is to find the path $r = \rho(\theta)$ that gives the minimum time τ for a particle of mass m—think of it as a bead sliding on a frictionless wire—starting at rest at r_0, θ_0 to travel to r_1, θ_1.

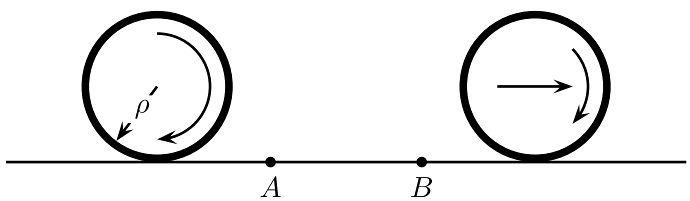


a) Write the travel time τ along a path $r = \rho(\theta)$ extending from r_0, θ_0 to r_1, θ_1 as a suitable integral over θ of an appropriate integrand. Make clear what you are doing.

b) What is the differential equation which the minimizing $\rho(\theta)$ must satisfy? As writing out the equation in full detail may be tedious, it will suffice if you clearly indicate how, starting from your answer to (a), you could obtain it.

c) Under what conditions would you expect to be able to use the solution to the usual brachistochrone problem in a uniform gravitational field g as an adequate approximation for the nonuniform field under consideration? Discuss.

4. A spinning disk of radius ρ, mass M, with initial angular speed ω_0 and horizontal speed zero, is gently lowered onto a horizontal table where because of friction between it and the table the disk begins to roll, and eventually after a time τ rolls without slipping with a constant angular speed ω_1 and a horizontal speed $v_1 = \rho\omega_1$ of the center of mass. See the figure. The table exerts normal and tangential forces f_n and f_t on the disk at the point of contact, while gravity exerts a force Mg downwards, which one can assume is applied to the center of the disk. For the purposes of this problem assume that all forces are in, and all motion takes place in, the x, y plane, x is horizontal and y is vertical, and the only components of torque and angular momentum that are of interest are those in the z direction, perpendicular to this plane. For an angular speed ω the angular momentum and kinetic energy of such a disk about its center (the center of mass) are given by $I\omega$ and $\frac{1}{2}I\omega^2$, respectively, where the moment of inertia is $I = cM\rho^2$, with $c \leq 1$ a constant.



a) Show that the total torque acting on the disk when evaluated about the contact point, or about any point on the table, such as points A and B in the figure, is zero. Show that the total angular momentum of the disk at a particular instant of time is the same if it is evaluated about the contact point or another point on the table, such as A or B.

b) Use the results in (a) to find the total angular speed ω_1 of the disk after it has started to roll without slipping, in terms of the initial ω_0, c, and possibly other quantities.

c) What is the total change in the kinetic energy of the cylinder from the time $t = 0$ when it is placed on the table until $t = \tau$ when it begins to roll without slipping? Is energy conserved? Why or why not?

5. A monoenergetic beam of alpha particles is directed at a thin carbon target. Assume that the alpha particles detected at a scattering angle of $\psi = 90°$ in the laboratory have a kinetic energy T_1 of 32 MeV, and the kinetic energy of the corresponding recoiling carbon nucleus is $T_2 = 12$ MeV. The carbon nucleus has a mass 3 times that of an alpha particle.

a) Use conservation laws to derive the formula $T_1'/T_2' = m_2/m_1$ for the ratio of the kinetic energies of the two particles in the center of mass.

b) Assuming an elastic collision, and making use of the result of (a), find the kinetic energy T_0 of the incident alpha particle in the laboratory before the collision, and the kinetic energies T_1' and T_2' of the alpha particle and carbon nucleus in the center of mass, for the scattering event described earlier.

c) Now suppose the collision is inelastic with a Q of -8 MeV (energy lost during the collision). How does this change your answers to (b) assuming, as before, final kinetic energies of 32 and 12 MeV for the scattered alpha particle and recoiling nucleus? Note that in the center of mass the kinetic energies will be different before and after the collision, and you should consider both situations.