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33-331 Physical Mechanics Fall 2008

Solutions to Final Exam

(1) Two masses m_1, m_2 on frictionless table

a) Lagrangian $L = T - U$ is given by

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$- \frac{1}{2} k \left[\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} - r_0 \right]^2$$

Solve $\vec{R} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$, $\vec{r} = \vec{r}_2 - \vec{r}_1$; $\vec{R} = (X, Y)$ $\vec{r} = (r, \theta)$

for $\vec{r}_1 = \vec{R} - \dfrac{m_2}{m_1 + m_2} \vec{r}$, $\vec{r}_2 = \vec{R} + \dfrac{m_1}{m_1 + m_2} \vec{r}$

$\dot{\vec{r}}_1 = \dot{\vec{R}} - \dfrac{m_2}{m_1 + m_2} \dot{\vec{r}}$, $\dot{\vec{r}}_2 = \dot{\vec{R}} + \dfrac{m_1}{m_1 + m_2} \dot{\vec{r}}$

$$\frac{1}{2} m_1 (\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1) + \frac{1}{2} m_2 (\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2) = \frac{1}{2} (m_1 + m_2) \dot{\vec{R}} \cdot \dot{\vec{R}}$$

$$+ \frac{1}{2} \left[\frac{m_1 m_2^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2}{(m_1 + m_2)^2} \right] \dot{\vec{r}} \cdot \dot{\vec{r}} + \underbrace{\left[\frac{-m_1 m_2}{m_1 + m_2} + \frac{m_1 m_2}{m_1 + m_2} \right]}_{0} \dot{\vec{R}} \cdot \dot{\vec{r}}$$

Therefore $T = \dfrac{1}{2} (m_1 + m_2) V^2 + \dfrac{1}{2} \mu (r^2 \dot{\theta}^2 + \dot{r}^2)$

where $V^2 = \dot{X}^2 + \dot{Y}^2 = \dot{R}^2$, $v^2 = \dot{\vec{r}} \cdot \dot{\vec{r}} = (r^2 \dot{\theta}^2 + \dot{r}^2)$, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

$$L = T - U = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} \mu (r^2 \dot{\theta}^2 + \dot{r}^2) - \frac{1}{2} k (r - r_0)^2$$

b) Using $p_j = \partial L/\partial \dot{q}_j$ we identify four momenta:

$$p_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X}\ ; \quad p_Y = \frac{\partial L}{\partial \dot{Y}} = M\dot{Y}\ ;$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r}\ ; \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$$

Since L is independent of X, Y, and θ, therefore p_X, p_Y, p_θ are conserved quantities: the components of total linear momentum, and the angular momentum about the center of mass. That they are constant follows from Lagrange's equations

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{d}{dt} p_j$$

$\partial L/\partial q_j = 0 \Longrightarrow p_j$ is constant.

In addition, since $\partial L/\partial t = 0$, the Hamiltonian $H = \sum_j p_j \dot{q}_j - L = T + U$ is a constant of the motion

c) The condition for circular orbits is that

$$V_{eff}(r) = U(r) + \frac{l^2}{2\mu r^2} = \frac{1}{2}k(r-r_0)^2 + \frac{l^2}{2\mu r^2}$$

should have a minimum for some choice of l,

Differentiate: $V'_{eff}(r) = k(r-r_0) - l^2/\mu r^3 = 0$

or $\boxed{k(r-r_0)r^3 = l^2/\mu}$. As $l^2 \geq 0$ there are no solutions for $r < r_0$ (r must be nonnegative). For $r = r_0$ there is a solution only if $l = 0$, but then the masses are stationary in the center of mass: no circular orbit. For any $r > r_0$ there will be some circular orbit, and since

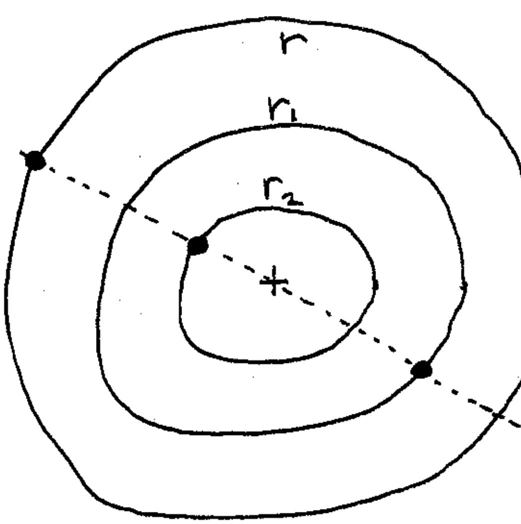
$$V''_{eff} = k + \frac{3l^2}{\mu r^4} > 0$$

, it is stable [Effective potential is a minimum]

For $m_1/m_2 = 1/2$, $\vec{R} = 0$ we have from (a)

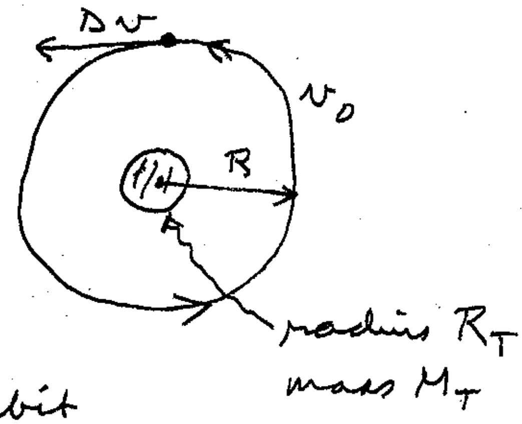
$$\vec{r}_1 = -\frac{2}{3}\vec{r}, \quad \vec{r}_2 = \frac{1}{3}\vec{r}$$

$\vec{r}_1(t) = -\frac{2}{3}\vec{r}(t)$, $\vec{r}_2(t) = \frac{1}{3}\vec{r}(t)$, so if $\vec{r}(t)$ traces out a circle the situation is as shown in the sketch at the right, where at a particular time t_0 $\vec{r}(t)$, $\vec{r}_1(t)$, $\vec{r}_2(t)$ are as indicated by the dots.



(2)

a) In order to escape from Thartus the kinetic energy of the spacecraft must be increased enough so that the total energy E is 0 (parabolic orbit) or $\frac{1}{2} m v_0^2$ if it is desired to achieve a hyperbolic orbit with speed v_0 far from Thartus. The best way of increasing the kinetic energy is to make Δv in the same direction as the instantaneous velocity on the circular orbit, as per the sketch, because the change in kinetic energy



$$\Delta T = \frac{1}{2} m (v_0 + \Delta v)^2 - \frac{1}{2} m v_0^2 = m\left(v_0 \Delta v + \frac{1}{2} \Delta v^2\right)$$

is then the greatest possible.

The initial energy in a circular orbit in a $1/r$ gravitational potential is

$$E_0 = T_0 + U_0 = -T_0$$

so to achieve a parabolic orbit one needs $\Delta T = T_0$, or $\frac{1}{2} m (v_0 + \Delta v)^2 = m v_0^2$ or $v_0 + \Delta v = \sqrt{2}\, v_0$, or $\Delta v = (\sqrt{2} - 1) v_0 = 0.414\, v_0$. But to achieve a hyperbolic orbit with speed v_0 far away, we need $\Delta T = 2 T_0$, which means $\frac{1}{2} m (v_0 + \Delta v)^2 = \frac{3}{2} m v_0^2$,

$$v_0 + \Delta v = \sqrt{3}\, v_0 \qquad \Delta v = (\sqrt{3} - 1)\, v_0 = 0.732\, v_0$$

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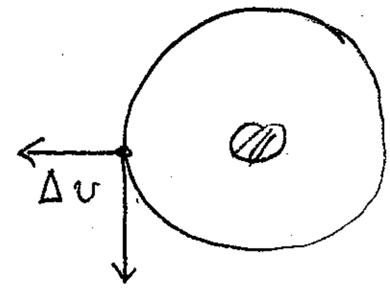
b) If the rocket is fired in a direction perpendicular to the one in (a), the new kinetic energy will be given by



$$T = \frac{1}{2} m \left(\vec{v}_0 + \Delta\vec{v}\right)^2$$

$$= \frac{1}{2} m v_0^2 + \frac{1}{2} m (\Delta v)^2$$

and $\Delta T = \frac{1}{2} m (\Delta v)^2$

Hence to achieve a parabolic orbit, $\Delta T = T_0$, one will need $\boxed{\Delta v = v_0}$ and to achieve a hyperbolic orbit with speed v_0 a great distance from Thartus will require $\Delta T = 2T_0$ and thus $\boxed{\Delta v = \sqrt{2}\, v_0 = 1.414\, v_0}$

c) ~~dt~~ The basic differential equation for rocket motion may be written as $m\,dv = -u\,dm$, or $dv = -\frac{u}{m}\,dm$. Integrating both sides we have

$v_1 - v_0 = -u\,(\ln m_1 - \ln m_0)$ where v_0 and v_1 are the initial and final velocities, and m_0 and m_1 the initial and final masses. Thus

$$\Delta v = v_1 - v_0 = u \ln (m_0/m_1)$$

In the case at hand we suppose $\Delta v = 3u$, and hence $\ln\left(\frac{m_0}{m_1}\right) = 3$ or $(m_0/m_1) = e^3$

$m_0 = m_S + m_R + m_F$ and $m_1 = m_S + m_R$, since the fuel have been exhausted. Hence

$$m_0/m_1 = 1 + \frac{m_F}{m_S + m_R} = e^3 \qquad \frac{m_F}{m_S + m_R} = e^3 - 1 = 19.$$

Ubrian technology is pretty good!

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3

Brachistochrone in 1/r potential

a) The potential energy of a particle of mass m in a gravitational potential $\Phi = -k/r$ is $m\Phi = U = -mk/r$. This plus the kinetic energy $\frac{1}{2}mv^2$ is constant, and since at $r = r_0$ the speed $v = 0$, we have $\frac{1}{2}mv^2 - mk/r = -mk/r_0$,

so $v^2 = r^2\dot\theta^2 + \dot r^2 = 2k\left(\frac{1}{r} - \frac{1}{r_0}\right)$. Along the curve $r = \rho(\theta)$ one has $\dot r = [\rho'(\theta)]\dot\theta$, thus $\boxed{\rho' = d\rho/d\theta}$

$$[\rho^2 + \rho'^2]\,\dot\theta^2 = 2k\left(\frac{1}{\rho} - \frac{1}{r_0}\right) = (\rho^2+\rho'^2)\left(\frac{d\theta}{dt}\right)^2$$

Therefore $$dt/d\theta = \sqrt{\frac{(\rho^2+\rho'^2)\,r_0\rho}{2k\,(r_0-\rho)}} =: f(\rho,\rho')$$

$$T = \int_0^{\theta_1} d\theta \sqrt{\frac{(\rho^2+\rho'^2)\,r_0\rho}{2k(r_0-\rho)}} = \int_0^{\theta_1} f(\rho,\rho')\,d\theta$$ where $\rho(\theta)$ is given and determines $\rho'(\theta)$

b) The differential equation for the minimizing $\rho(\theta)$ is obtained from Euler's equation $\frac{\partial f}{\partial \rho} = \frac{d}{d\theta}\frac{\partial f}{\partial \rho'}$ or, since $\frac{\partial f}{\partial\theta} = 0$, one can also use $f - \rho'\frac{\partial f}{\partial\rho'} = \text{constant} = K$

$$\frac{\partial f}{\partial\rho'} = \rho'\sqrt{\frac{r_0\rho}{2k(r_0-\rho)(\rho^2+\rho'^2)}}$$

So the differential equation, or its first integral, can be written as $$\rho'^2\sqrt{\frac{r_0\rho}{2k(r_0-\rho)(\rho^2+\rho'^2)}} = \sqrt{\frac{(\rho^2+\rho'^2)\,r_0\rho}{2k\,(r_0-\rho)}} - K$$

c) One would expect to be able to use the solution of the brachistochrone problem for uniform gravity in cases in which any plausible $r = \rho(\theta)$ remains in a domain where the gravitational potential is approximately of the form

$$\Phi(\vec{r}) \simeq \text{const} + gz + b\left[\frac{x^2+y^2}{2} - z^2\right] + \cdots$$

where x, y, z are measured from the point $\vec{r} = (r_0, \theta = 0)$ in that $z = r - r_0$, $x = r_0\theta$ and we set $y = 0$ to lowest order in r and θ. Here b will be of order g/r_0, so if $|z| \ll r_0$, say $z = \epsilon r_0$, $|\epsilon| \ll 1$, ~~the~~ and likewise x is of the same order — which means $|\theta| \ll 1$, the 'b' terms will be small compared with gz. Consequently, we expect that when

$$r_0 - r_1 \ll r_0, \qquad \theta_1 \ll 1$$

the solution to the uniform-field brachistochrone problem should be adequate for the case under consideration.

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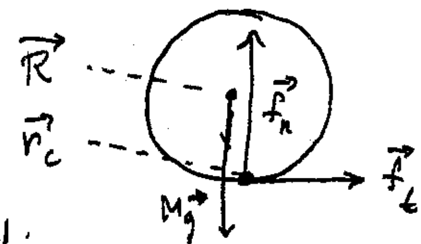
(4)

Spinning disk on ~~flat~~ table

a) Let $\vec{R}$ be the center of mass = center of disk, $\vec{r}_c$ be the point of contact, $M\vec{g}$ force of gravity, $\vec{f}_n$ and $\vec{f}_t$ are normal and tagential forces of table on disk. For clarity $M\vec{g}$ and $\vec{f}_n$ have been slightly distorted; $\vec{f}_n$ passes through $\vec{R}$, $M\vec{g}$ through $\vec{r}_c$. The total torque on the disk will be $\vec{N} = \vec{R} \times M\vec{g} + \vec{r}_c \times \vec{f}_n + \vec{r}_c \times \vec{f}_t$



As the disk has zero net momentum in the upwards direction, necessarily $M\vec{g} = -\vec{f}_n$, so
$\vec{R} \times M\vec{g} + \vec{r}_c \times \vec{f}_n = (\vec{R} - \vec{r}_c) \times M\vec{g} = 0$ because $\vec{R} - \vec{r}_c$ is parallel to $\vec{g}$. But if we compute the torque about a point on the table, $\vec{r}_c$ will be a vector parallel to $\vec{f}_t$, so $\vec{r}_c \times \vec{f}_t = 0$; about the contact point itself $\vec{r}_c = 0$. So $\vec{N} = 0$ about a point on the table, including the contact point.

The total angular momentum $\vec{L} = \vec{R} \times M\vec{V} + \vec{L}'$, where $\vec{L}'$ = angular momentum about center of mass, $\vec{V}$ = velocity of center of mass. Since $\vec{V}$ is parallel to the table, if $\vec{R}$ is measured from any point on the table only the component perpendicular to $\vec{V}$ enters $\vec{R} \times \vec{V}$, i.e., $\vec{R} \times \vec{V} = (\vec{R} - \vec{r}_c) \times \vec{V}$ is independent of which point on the table is the origin of $\vec{R}$. Obviously $\vec{L}'$ does not depend on the origin of coordinates, so $\vec{L}$ is the same about any point on the table top.

b) About some point on the table top we know that $\vec{L}$ must be independent of time, since $\dot{\vec{L}} = \vec{N}$ and $\vec{N} = 0$, by part (a). Initially $L = I\omega_0$, and after the disk has stopped slipping

$$L = M\rho V + I\omega_1 = M\rho v_1 + I\omega_1$$
$$= M\rho^2\omega_1 + cM\rho^2\omega_1 = (1+c)M\rho^2\omega_1$$

Since $I\omega_0 = cM\rho^2\omega_0$, we conclude that

$$cM\rho^2\omega_0 = (1+c)M\rho^2\omega_1 \Rightarrow \boxed{\omega_1/\omega_0 = \frac{c}{1+c}}$$

c) The initial kinetic energy of the cylinder is

$$T_0 = \frac{1}{2}I\omega_0^2 = \frac{1}{2}cM\rho^2\omega_0^2$$

Its final kinetic energy has two terms; rotation about the center of mass plus motion of the center of mass:

$$T_1 = \frac{1}{2}I\omega_1^2 + \frac{1}{2}MV^2 = \frac{1}{2}M\rho^2(1+c)\omega_1^2$$
$$= \frac{1}{2}M\rho^2\omega_0^2\left[\frac{c^2}{1+c}\right]$$

So $T_0 - T_1 = \frac{1}{2}M\rho^2\omega_0^2\left[c - \frac{c^2}{1+c}\right] = \frac{1}{2}\left(\frac{c}{1+c}\right)M\rho^2\omega_0^2$

is the change in kinetic energy, which decreases unless $c = 0$. So energy is not conserved, and this is not surprising, because there is frictional sliding going on for $t < T$, and kinetic energy is transformed to heat.

(5)

Alpha particles on carbon target

a) Derive $T_1'/T_2' = m_2/m_1$ for center of mass.

In the center of mass system the center of mass is (by definition) not moving, so the total momentum is

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

where $\vec{v}_1$ and $\vec{v}_2$ are the velocities of particles 1 and 2. This means their magnitudes $|\vec{v}_j| = v_j$ are related by

$$v_1 = (m_2/m_1)\, v_2$$

The kinetic energies are thus related by

$$\frac{T_1'}{T_2'} = \frac{\frac{1}{2}m_1 v_1^2}{\frac{1}{2}m_2 v_2^2} = \frac{m_1}{m_2}\left(\frac{m_2}{m_1}\right)^2 \frac{v_2^2}{v_2^2} = m_2/m_1$$

b) Kinetic energy of alpha before collision $= T_0 =$ sum of kinetic energies after collision (collision elastic $\Rightarrow Q=0$) $= 32+12 = 44$ MeV, by conservation of energy. But then

$$T_0 = \tfrac{1}{2}MV^2 + T_1' + T_2'$$ = center of mass energy + energy about the center of mass.

Now $V = \dfrac{m_1 u_1}{m_1+m_2}$ = velocity of center of mass, u_1 = velocity of incident α.

Thus $V = \dfrac{1}{1+3}u_1 = \dfrac{1}{4}u_1$, $MV^2 = (m_1+m_2)V^2 = \dfrac{4}{4^2} m_1 u_1^2$

so $\frac{1}{2}MV^2 = \frac{1}{4}\left(\frac{1}{2}m_1 u_1^2\right) = 11$ MeV, and

$T_1' + T_2' = 44 - 11 = 33$ MeV, while $T_1'/T_2' = 3$, so

$$\boxed{T_1' = 24\tfrac{3}{4}\text{ MeV},\ T_2' = 8\tfrac{1}{4}\text{ MeV}}$$

c) Given a $Q = -8$ MeV, with final energies 32 and 12 MeV for alpha and carbon, the initial energy of the alpha must have been

$$\boxed{T_0 = 32 + 12 + 8 = 52 \text{ MeV}}$$

The center of mass term $\frac{1}{2}MV^2$ is $\frac{1}{4}$ of this or 13 MeV, using same argument as in (b), so in the center of mass before the collision we have

$$52 - 13 = 39 \text{ MeV} = T_1' + T_2'$$

The ratio $T_1'/T_2' = 3$ holds before or after the collision, so now $\boxed{T_1' = 29\,^1/_4 \text{ MeV},\ T_2' = 9\frac{3}{4} \text{ MeV}}$ before collision

But after the collision takes place we have

$T_1'' + T_2'' = 39 - 8 = 31$ MeV, so splitting this up using $T_1''/T_2'' = 3$ we get

$$\boxed{T_1'' = 23\frac{1}{4} \text{ MeV},\ T_2'' = 7\frac{3}{4} \text{ MeV}}$$ after collision