33-331 Physical Mechanics I. Fall Semester, 2009
Assignment No. 8
Due Friday, October 30

READING

Thornton and Marion Ch. 8, Secs. 8.1 through 8.6

READING AHEAD:

Thornton and Marion Ch. 8, Secs. 8.7, 8.8
Handout: Hyperbolic orbits

EXERCISES

1. Turn in at most one page, and not less than a third of a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc.

2. Let the Hamiltonian of a particle in three dimensions be given by

$$H = \frac{1}{2m}\left[(p_x - \alpha y)^2 + p_y^2 + p_z^2\right],$$

where p_x, p_y, p_z are the momenta conjugate to the Cartesian coordinates x, y, z, and $\alpha > 0$ is a constant.

a) What can you say about conserved quantitities by simple inspection of H?

b) Find the Hamilton equations of motion.

c) Use the results in (b) to obtain expressions for the different components of acceleration $(\ddot{x}, \ddot{y}, \ddot{z})$ in terms of the Cartesian coordinates and/or their first time derivatives. [Hint. You only need the time derivatives, and the equations are quite simple.]

d) Find a general solution to the equations in (c) with the expected 6 free parameters. (There are six because you can prescribe the initial position and velocity of the particle. However, you may prefer to parametrize solutions in a different way.)

e) Comment about the conserved quantitites you noted in (a) in light of your solution to (d). Might there be something else that is conserved?

f) Find the Lagrangian L corresponding to H, expresed as a function of the standard variables (x, $\dot{x}$, etc.)

3. Sketch a typical phase-space trajectory as a function of time for a particle confined to a one-dimensional box with walls at $x = -a, +a$, assuming it undergoes elastic collsions at the walls. Then discuss what happens as a function of time to a little region initially inside a samll rectangle $0 < p_1 \leq p \leq p_1 + \Delta p$, $-\Delta x/2 \leq x \leq \Delta x/2$. Show that for short times its area remains constant. (Its area remains constant at all times, but after a while its shape gets rather complicated.)

4. The Lagrangian of a one-dimensional harmonic oscillator is $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. Show that by re-expressing this in terms of $q = \beta x$ and $\dot{q}$, where β is a suitable positive constant depending on m and k, it is possible to have a Hamiltonian of the form $C(p^2 + q^2)$, with p $(=\partial L/\dot{q})$ the momentum conjugate to q. Show that in this case the trajectories in phase space are circles. What happens to the region bounded by the lines $\theta = \theta_1$, $\theta = \theta_1 + \Delta\theta$ and $r = r_1$, $r = r_1 + \Delta r$ in polar coordinates? Show that this is consistent with Liouville.

5. A particle is moving in a central force. Initially its radial distance is R_1 and its radial velocity $\dot{r}$ is equal to its azimuthal velocity $r\dot{\theta}$. It spirals outwards until at a much later time it is moving in an almost circular orbit a distance R_2 from the center. Find an expression for the *change* in its potential energy $\Delta U = U(R_2) - U(R_1)$ in terms of its initial kinetic energy T_1 at the time when $r = R_1$ and the ratio R_2/R_1.