<u>Problem Solutions:</u> Set 8 (October 22, 2003)

33. a) $V(x)$ has minima, at $x = 0$ and $x = a$, and a maximum at $x = a/2$.

b) $F(x) = -\dfrac{dV}{dx}$. This can be written in several different useful forms:

$$F(x) = -kx\left(1 - \frac{x}{a}\right)\left(1 - \frac{2x}{a}\right) = -kx\left(1 - \frac{3x}{a} + \frac{2x^2}{a^2}\right) = -kx + \frac{3kx^2}{a} - \frac{2kx^3}{a^2}.$$

(The first displays the roots explicitly; the third is probably most convenient for numerical computations.)

Equilibrium points occur where $F = 0$, i.e., at $x = 0$, $x = \dfrac{a}{2}$, $x = a$.

Substitute into $V(x)$ to find $V(0) = V(a) = 0$, $V(a/2) = \dfrac{ka^2}{32}$.

From graph, or from second-derivative test, $x = 0$ and $x = a$ are minima of V, and thus *stable* equilibrium points; $x = a/2$ is a maximum and an *unstable* equilibrium point.

c) At $x = 0$, $F(x) = -kx + \dfrac{3k}{a}x^2 - \dfrac{2k}{a^2}x^3$.

At $x = a$, $F(x) = -k(x-a) - \dfrac{3k}{a}(x-a)^2 - \dfrac{2k}{a^2}(x-a)^3$.

In each case, the effective force constant is just k, and the frequency of small oscillations about each point is $\omega_o = \sqrt{\dfrac{k}{m}}$. If $k = 1$ and $m = 1$, then $\omega_o = 1$ and the period is $T = \dfrac{2\pi}{\omega_o} = 2\pi \cong 6.283$.

d) $$-kx + \frac{3kx^2}{a} - \frac{2kx^3}{a^2} = m\frac{d^2x}{dt^2}.$$ If $k = 1$, $m = 1$, $a = 1$, Maple equation is

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diffeq := diff(x(t), t$2) = -x(t) + 3*x(t)^2 - 2*x(t)^3;
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e) Period: (from x vs. t graphs)

v_o	0.01	0.05	0.10	0.20
T	6.3	6.4	6.5	7.6

Phase trajectory is nearly elliptical when $v_o = 0.01$ and becomes increasingly distorted for larger values of v_o.

(continued)

33. (continued)

f) Use conservation of energy; total energy at $x = 0$ is equal to that at $x = a/2$:

$$\frac{1}{2} m v_o^2 + 0 = 0 + \frac{ka^2}{32}, \qquad v_o = \frac{a}{4}\sqrt{\frac{k}{m}} = \frac{a}{4}\,\omega_o \quad (!)$$

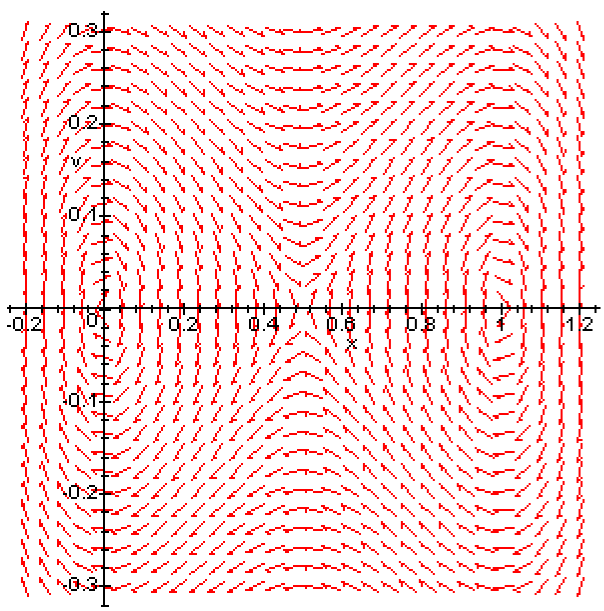
If $k = 1$, $m = 1$, $a = 1$, then the critical value of v_o is 1/4. When $v_o > 0.250$, the character of the motion changes completely, and when v_o is close to 0.250 the motion is extremely sensitive to small changes in v_o.

Dividend: When $v_o = \frac{a}{4}\sqrt{\frac{k}{m}}$, the limits of motion are $x = \frac{1 \pm \sqrt{2}}{2}\, a$,

or approximately $x = -0.207a$, $+1.207a$.

g) When $v_o = 1$, motion is symmetric about $x = 0.5$. Phase plot is pumpkin shaped, with dips on the top and bottom sections near $x = 0.5$, corresponding to particle slowing down near the maximum in $V(x)$ (at $x = 1/2$).

This field plot was made using x = –0.2..1.2, v = –0.3..0.3, and dirgrid = [30,30]. A plot with x = –0.2..0.6 shows only one of the two attractors, but makes the phase trajectories more visible.



34. a) $$-kx + \frac{3kx^2}{a} - \frac{2kx^3}{a^2} - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}.$$

If $k = 1$, $m = 1$, $a = 1$, $b = 0.1$, Maple equation is

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diffeq := diff(x(t), t$2) = -x(t) + 3*x(t)^2 - 2*x(t)^3 - (0.1)*diff(x(t), t);
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b) Oscillations near $x = 1$ are mirror images of oscillations near $x = 0$ (with the mirror at $x = 1/2$).

c) When $v_o > 0.2813$, motion is no longer confined to one of the two wells of $V(x)$. This critical value is somewhat larger than the value v_o found in Problem 33(f) because energy is dissipated by the damping forces.

d) The crossover occurs at about $v_o = 1.0929$. Similar crossovers occur at about $v_o = 0.2813$, 0.4062, 0.5523, 0.7162, and 0.8965, where the particle makes increasing numbers of transits across both wells before settling into one well or the other.