Problem Solutions: Set 7 (October 15, 2003)

30. a) $\langle \sin^2 \omega t \rangle = \dfrac{1}{T}\int_0^T \sin^2 \omega t \, dt$; $\langle \cos^2 \omega t \rangle = \dfrac{1}{T}\int_0^T \cos^2 \omega t \, dt$,

$\langle \sin \omega t \cos \omega t \rangle = \dfrac{1}{T}\int_0^T \sin \omega t \cos \omega t \, dt$. Substitute $\omega = 2\pi/T$, then evaluate the integrals using Maple.

c) In small-damping approximation,

$$x = \frac{F_o/2m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}} \cos(\omega t + \varphi), \quad v = -\frac{\omega F_o/2m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}} \sin(\omega t + \varphi),$$

$$P_{out} = -bv^2 = -\frac{(2m\gamma)(\omega F_o/2m\omega_o)^2}{(\omega_o - \omega)^2 + \gamma^2} \sin^2(\omega t + \varphi).$$

The average value of $\sin^2(\omega t + \varphi)$ over a cycle is 1/2, and P_{out} is appreciably different from zero only when $\omega \cong \omega_o$, and we get

$$\langle P_{out} \rangle = -\frac{\gamma F_o^2/4m}{(\omega_o - \omega)^2 + \gamma^2}.$$

This is greatest when the denominator is smallest, i.e., when $\omega = \omega_o$. Then

$$\langle P_{in} \rangle = \frac{F_o^2}{4m\gamma}.$$

d) Both F and v are proportional to F_o, amplitude is greatest when $\omega = \omega_o$.

e) From (c), $v = -\dfrac{\omega F_o/2m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}} \sin(\omega t + \varphi)$. Also, $F = F_o \cos \omega t$.

$$P_{in} = Fv = -(F_o \cos \omega t)\frac{\omega F_o/2m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}}(\sin \omega t \cos \varphi + \cos \omega t \sin \varphi).$$

The average value of $\sin \omega t \; \cos \omega t$ over a cycle is zero, and the average value of $\cos^2 \omega t$ is 1/2.

$$\langle P_{in} \rangle = -\frac{\omega F_o^2/4m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}} \sin\varphi.$$

(continued)

30. (continued)

From class discussion and small damping approximation (when ω is near ω_o,)

$$\sin\varphi = \frac{-2\gamma\omega}{\sqrt{\left(\omega^2 - {\omega_o}^2\right)^2 + (2\gamma\omega)^2}} \cong \frac{-2\gamma\omega}{\omega_o\sqrt{(\omega_o - \omega)^2 + \gamma^2}}, \qquad \text{and}$$

$$\langle P_{in}\rangle = -\frac{\omega {F_o}^2/4m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}}\sin\varphi = -\frac{\omega {F_o}^2/4m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}}\,\frac{-2\gamma\omega}{\omega_o\sqrt{(\omega_o - \omega)^2 + \gamma^2}}.$$

When ω is close to ω_o,

$$\langle P_{in}\rangle = \frac{\gamma {F_o}^2/4m}{(\omega - \omega_o)^2 + \gamma^2}.$$

Note that this is equal, apart from sign, to $\langle P_{out}\rangle$ as found in (c). Thus this result directly verifies conservation of energy; $\langle P_{in}\rangle + \langle P_{out}\rangle = 0$.

$\langle P_{in}\rangle$ is greatest when the denominator is smallest (at $\omega = \omega_o$). The maximum value is

$$\langle P_{in}\rangle_{max} = \frac{{F_o}^2}{4m\gamma}.$$

31. a) $$x(t) = \frac{F_o/m}{\sqrt{(\omega^2 - {\omega_o}^2)^2 + (2\gamma\omega)^2}}\cos(\omega t + \varphi),$$

$$v(t) = \frac{-\omega F_o/m}{\sqrt{(\omega^2 - {\omega_o}^2)^2 + (2\gamma\omega)^2}}\sin(\omega t + \varphi), \quad \text{so} \quad v_{max} = \frac{\omega F_o/m}{\sqrt{(\omega^2 - {\omega_o}^2)^2 + (2\gamma\omega)^2}}.$$

b) Use Maple to calculate $\dfrac{dv_{max}}{d\omega}$, set the result equal to zero, and solve for ω.

Result: v_{max} is greatest when $\omega = \omega_o$, and the value of v_{max} is then $v_{max} = \dfrac{F_o}{2m\gamma}$.

32. a) When $|x| < 1$, the velocity-dependent force has the same direction as v, so it adds energy to the system. When $|x| > 1$, energy is dissipated. Hence the motion never damps down to $x = 0$; instead, it tends toward some sort of limit cycle.

b) The limit cycle has a period of approximately 6.6 s. The amplitude is about 2.0 m, and the maximum velocity is about 2.6 m/s. The limit cycle is independent of the initial conditions; a trajectory that starts at a large value of x or v will spiral in to smaller values, while a trajectory that starts at a small value of x or v spirals out to larger values. The same limiting trajectory (limit cycle) is approached in each case. The final state of motion is periodic and is independent of initial conditions. This limiting trajectory is also called an *attractor*. For a damped oscillator with damping force $F = -bv$, the attractor is the origin in phase space, i.e., the point $x = 0$, $v = 0$.

c) $$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -x + (1 - x^2)\,v.$$

```
restart;
eq1 := diff(x(t), t) = v(t);
eq2 := diff(v(t), t) = -x(t) + (1 - x(t)^2)*v(t);
with(DEtools, dfieldplot);
dfieldplot({eq1, eq2}, [x(t),v(t)],  t = 0..10, x = -3..3, v = -3..3,
            dirgrid = [30, 30], color = black);
```

d) From the vector field plot, if the phase point is initially at $x = 0$, $v = 0$ and is displaced in any direction (by giving the particle a small displacement or velocity), the phase point tends to move away from the origin (unlike, for example, the damped harmonic oscillator, where all the phase trajectories spiral in toward the origin in phase space).

This same conclusion also follows from looking at the equations in (c), starting at $x = 0$, $v = 0$, then giving x or v a small increment and observing where the phase point goes.