**Problem Solutions:** Set 3 (September 17, 2003)

11. a) $\dfrac{dx}{x} = -dt$, $\ln x = -t + \text{constant}$, $x = (\text{constant})e^{-t}$.

$1 = (\text{constant})e^{-0}$, $x = e^{-t}$.

b) At $t = 1$, $e^{-t} = 0.3679$; Euler method gives 0.3487 (5.2% too small).

c) At $t = 1$, $e^{-t} = 0.3679$; Euler method gives 0.3639 (1.1% too small).

Note that reducing step size by 1/5 reduces error by about 1/5.

12. a) $F = bv^2 = b\left(v_x^{\,2} + v_y^{\,2}\right)$.

$$F_x = -\frac{v_x}{v}F = -bv_x v = -b\left(\frac{dx}{dt}\right)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2},$$

$$F_y = -b\left(\frac{dy}{dt}\right)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2}.$$

$$F_x = m\frac{d^2x}{dt^2}, \quad F_y - mg = m\frac{d^2y}{dt^2}.$$

b) diffeqx := –b*diff(x(t), t)*sqrt(diff(x(t), t)^2 + diff(y(t), t)^2)
= m*diff(x(t), t$2):

diffeqy := –b*diff(y(t), t)*sqrt(diff(x(t), t)^2 + diff(y(t), t)^2) – m*g
= m*diff(y(t), t$2)

init1 := x(0) = 0; init2 := y(0) = 0; init3 := D(x)(0) = vxo; D(y)(0) = vyo;

c) vx0 := 10; vyo := 20; b := 0;
solution := dsolve({diffeqx, diffeqy, init1, init2, init3, init4},
{x(t), y(t)});
x := rhs(solution[1]; y := rhs(solution[2]; (May be in opposite order.)

$y_{\max} = 20$ m, $x_{\max} = 40$ m, $t_{\rm tot} = 4$ s.

(continued)

12. (continued)

d) Continue code from part (c):

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b := 2;   m := 80;
solution := dsolve({diffeqx, diffeqy, init1, init2, init3, init4}, {x(t), y(t)},
     numeric);
with(plots, odeplot);
odeplot(solution, [x(t), y(t)], 0..4);
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13. Assume air resistance force is proportional to $v^2$; use results from Problem 12. Use Problem 10 to determine the constant $b$. For a smooth sphere, $D = 0.44$; for a somewhat rough sphere, guess $D = 0.5$. Convert parameters to SI units:

$m = 0.145$ kg, $r = 0.0369$ m, $A = \pi r^2 = 0.00427\ \text{m}^2$.

Pittsburgh: $b = \frac{1}{2}\rho A D = \frac{1}{2}(1.18\ \text{kg/m}^3)(0.00427\ \text{m}^2)(0.5) = 0.00126\ \text{kg/m}$;

Denver: $b = 0.00105$ kg/m.

Reasonable guess for initial velocity components: $v_{xo} = 40$ m/s, $v_{yo} = 30$ m/s.
Then $v = 50$ m/s ( = 112 mi/hr).

Using these values with Maple analysis in Problem 12, we find the following ranges:

Pittsburgh: 107 m = 351 ft; Denver: 117 m = 384 ft; if $b = 0$, 245 m = 803 ft.

Conclusion: air resistance is a significant effect. The Denver ball goes over 30 feet farther than the Pittsburgh ball, and if there were no air resistance the ball would go more than twice as far, given the same initial velocity.

Note: The variation of $g$ with elevation is a negligible effect, less than 0.1 %. Can you verify this, using Newton's law of gravitation?

14. a) $F(x) = -\dfrac{dV}{dx} = V_o\left(\dfrac{2a^2}{x^3} - \dfrac{a}{x^2}\right).$

b) $F(x) = 0$ when $x = 2a.$

$$\frac{d^2V}{dx^2} = V_o\left(\frac{6a^2}{x^4} - \frac{2a}{x^3}\right); \qquad \text{when} \qquad x = 2a, \quad \frac{d^2V}{dx^2} = \frac{V_o}{8a^2}.$$

So $x = 2a$ is a point of stable equilibrium (and the only one).

d) $x_{min} = 2.764$, $x_{max} = 7.236$

e) Kinetic energy must be greater than depth of potential well, so minimum value of $v_o$ (the "escape velocity") is given by

$$\tfrac{1}{2} m v_o^2 - \frac{V_o}{4} = 0, \qquad \text{or} \qquad v_o = \sqrt{\frac{V_o}{2m}} = \sqrt{\frac{5.00\ \text{J}}{2(0.500)\ \text{kg}}} = 2.236\ \text{m/s}.$$