**Problem Solutions:** Set 14 (December 1, 2003)

55. a) With Maple:

```
restart;
y := A*sin(n*Pi*x/L)*sin(n*Pi*c*t/L);
c := sqrt(F/mu);
assume(n, integer);
E := int((mu/2)*diff(y,t)^2 + (F/2)*diff(y,x)^2, x = 0..L);
simplify(E);
```

Without Maple:

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{L} A \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}, \qquad \frac{\partial y}{\partial x} = \frac{n\pi}{L} A \cos\frac{n\pi x}{L} \sin\frac{n\pi ct}{L}.$$

$$E = \frac{A^2}{2}\int_0^L \left[\mu \frac{n^2\pi^2c^2}{L^2} \sin^2\frac{n\pi x}{L}\cos^2\frac{n\pi ct}{L} + F\frac{n^2\pi^2}{L^2}\cos^2\frac{n\pi x}{L}\sin^2\frac{n\pi ct}{L}\right] dx$$

$$\mu c^2 = F \quad \text{and} \quad \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \int_0^L \cos^2\frac{n\pi x}{L}\,dx = \frac{L}{2}, \qquad \text{so}$$

$$E = \frac{\pi^2 F}{4L} n^2 A^2\left[\cos^2\frac{n\pi ct}{L} + \sin^2\frac{n\pi ct}{L}\right] = \frac{\pi^2 F}{4L} n^2 A^2 .$$

b)
```
restart;
y1 := A1*sin(n1*Pi*x/L)*sin(n1*Pi*c*t/L);
y2 := A2*sin(n2*Pi*x/L)*sin(n2*Pi*c*t/L);
y := y1 + y2;
c := sqrt(F/mu);
assume(n1, integer, n2, integer);
E := (1/2)*int(mu*diff(y,t)^2 + F*diff(y,x)^2, x = 0..L);
```

Final result:

$$E = \frac{\pi^2 F}{4L}\left(n_1{}^2 A_1{}^2 + n_2{}^2 A_2{}^2\right) = \frac{\pi^2 F}{4L}\sum_i n_i{}^2 A_i{}^2 \qquad \text{(for any number of modes).}$$

56. a) $y(x,t) = A \sin kx \sin \omega t$;

$$P = -F\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -F(kA\cos kx \sin \omega t)(\omega A \sin kx \cos \omega t)$$

$$= \frac{F\omega k A^2}{4}\sin(2kx)\sin(2\omega t) \qquad \text{(from double - angle identity).}$$

Also, $k = \frac{\omega}{c} = \omega\sqrt{\frac{\mu}{F}}$, so $P = \frac{\sqrt{\mu F}}{4}\omega^2 A^2 \sin(2kx)\sin(2\omega t)$.

b) $P_{max} = \frac{\sqrt{\mu F}}{4}\omega^2 A^2$.

$P_{max}$ occurs at points for which $2kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$ and

$x = \frac{\pi}{4k}, \frac{3\pi}{4k}, \frac{5\pi}{4k}, \cdots$. These points are midway between each node and the adjacent antinodes.

c) From above, $P = 0$ when $2kx = n\pi$ and $x = \frac{n\pi}{2k}$.

These are the positions of the nodes and antinodes. When $\sin(2\omega t)$ is zero, the antinodes are all at $y = 0$, moving with maximum speed; the rope is instantaneously aligned with the $x$ axis. The energy is entirely kinetic, concentrated near the antinodes. When $\sin(2\omega t) = \pm 1$, the antinodes are at maximum displacement and instantaneously at rest. The slope is greatest near the nodes, the energy is entirely potential and concentrated near the nodes. Thus there is a constant flow of energy back and forth, twice each cycle, between each node and its two neighboring antinodes.

57. a) Substitute $\Psi$ into wave equation: $\frac{\partial^2\Psi}{\partial x^2} = -k^2\Psi$ and $\frac{\partial\Psi}{\partial t} = -i\omega\Psi$;

$\frac{-\hbar^2}{2m}(-k^2\Psi) = i\hbar(-i\omega\Psi)$. $\Psi$ is a solution if $\frac{\hbar^2 k^2}{2m} = \hbar\omega$.

If $p = \hbar k$ and $E = \hbar\omega$, then $E = \frac{p^2}{2m}$, consistent with classical energy-momentum relation.

b) Phase velocity is $v = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{p}{2m}$ (not $\frac{p}{m}$ as expected).

c) $\omega = \frac{\hbar k^2}{2m}$; group velocity is $v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m}$ (as expected).