33-231 Physical Analysis Fall 2003

Problem Solutions: Set 10 (November 5, 2003)

39. a)

$$k(x_2 - x_1) = m\ddot{x}_1,$$
$$k(x_3 - x_2) - k(x_2 - x_1) = m\ddot{x}_2,$$
$$-k(x_3 - x_2) = m\ddot{x}_3.$$

b) Plug in $x_1 = a_1 \cos \omega t$, etc., and re-arrange, letting $\lambda = \omega^2$.

$$(k - \lambda m)a_1 - ka_2 = 0,$$
$$-ka_1 + (2k - \lambda m)a_2 - ka_3 = 0,$$
$$-ka_2 + (k - \lambda m)a_3 = 0.$$

This system has non-trivial solutions if and only if its determinant is zero:

$$\begin{vmatrix} k - \lambda m & -k & 0 \\ -k & 2k - \lambda m & -k \\ 0 & -k & k - \lambda m \end{vmatrix} = 0, \quad (k - \lambda m)^2(2k - \lambda m) - 2k^2(k - \lambda m) = 0.$$

Factor out $(k - \lambda m)$, multiply out, re-arrange, and factor again, to obtain:

$$m\lambda(k - \lambda m)(3k - \lambda m) = 0, \quad \text{and} \quad \lambda_1 = 0, \quad \lambda_2 = \frac{k}{m}, \quad \lambda_3 = \frac{3k}{m}.$$

$$\lambda_1: \quad a_1 = a_2 = a_3.$$
$$\lambda_2: \quad a_1 = -a_3, \quad a_2 = 0.$$
$$\lambda_3: \quad a_1 = a_3, \quad a_2 = -2a_1.$$

In the first mode, all three masses move with constant velocity (zero frequency). In the second mode, the middle mass is stationary and the two end masses vibrate with equal amplitudes and a half-cycle phase difference, with frequency $\omega_2 = \sqrt{k/m}$. In the third, the end masses vibrate in phase with equal amplitude, and the center mass moves half a cycle out of phase with twice that amplitude.

c)

$$x_1 = q_1 + q_2 + q_3,$$
$$x_2 = q_1 - 2q_3,$$
$$x_3 = q_1 - q_2 + q_3.$$

Substitute into equations of motion:

$$-kq_2 - 3kq_3 = m(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3),$$
$$6kq_3 = m(\ddot{q}_1 - 2\ddot{q}_3),$$
$$kq_2 - 3kq_3 = m(\ddot{q}_1 - \ddot{q}_2 + \ddot{q}_3).$$

(continued)

39. c) (continued)

Adding the three equations gives $3m\ddot{q}_1 = 0, \qquad \lambda_1 = \omega_1^{\,2} = 0.$

The first minus the third gives: $-2kq_2 = 2m\ddot{q}_2, \qquad \lambda_2 = \omega_2^{\,2} = \frac{k}{m}.$

First plus third minus twice second: $-18kq_3 = 6m\ddot{q}_3, \qquad \lambda_3 = \omega_3^{\,2} = \frac{3k}{m}.$

d) $$E = \tfrac{1}{2}m(\dot{x}_1^{\,2} + \dot{x}_2^{\,2} + \dot{x}_3^{\,2}) + \tfrac{1}{2}k(x_2 - x_1)^2 + \tfrac{1}{2}k(x_3 - x_2)^2.$$

Substitute normal coordinate transformation from (c), and simplify:

$$E = \tfrac{3}{2}m\dot{q}_1^{\,2} + (m\dot{q}_2^{\,2} + kq_2^{\,2}) + (3m\dot{q}_3^{\,2} + kq_3^{\,2}).$$

e)

$\mathbf{x} = \mathbf{A}\mathbf{q}$: $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

$\mathbf{q} = \mathbf{A}^{-1}\mathbf{x}$: $$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

Check: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$

40. a) $-kx_1 + k'(x_2 - x_1) + F_o \cos\omega t = m\ddot{x}_1,$

$-kx_2 - k'(x_2 - x_1) = m\ddot{x}_2.$

Substitute $x_1 = a_1 \cos\omega t,\quad x_2 = a_2 \cos\omega t;$ collect terms and simplify:

$$\left(k + k' - \omega^2 m\right)a_1 - k' a_2 = F_o,$$
$$-k' a_1 + \left(k + k' - \omega^2 m\right)a_2 = 0.$$

Note that these equations are *not* homogeneous, and the determinant of the system is *not* zero. Now a_1 and a_2 are multiples of F_o. Solve using determinants:

$$a_1 = \frac{\begin{vmatrix} F_o & -k' \\ 0 & k + k' - \omega^2 m \end{vmatrix}}{\begin{vmatrix} k + k' - \omega^2 m & -k' \\ -k' & k + k' - \omega^2 m \end{vmatrix}} = \frac{\left(k + k' - \omega^2 m\right)F_o}{\left(k + k' - \omega^2 m\right)^2 - k'^2},$$

$$a_2 = \frac{\begin{vmatrix} k + k' - \omega^2 m & F_o \\ -k' & 0 \end{vmatrix}}{\begin{vmatrix} k + k' - \omega^2 m & -k' \\ -k' & k + k' - \omega^2 m \end{vmatrix}} = \frac{k' F_o}{\left(k + k' - \omega^2 m\right)^2 - k'^2}.$$

Simple numerical example: Take $F_o = m = k = k' = 1.$ Then

$$a_1 = \frac{2 - \omega^2}{\left(2 - \omega^2\right)^2 - 1}, \qquad a_2 = \frac{1}{\left(2 - \omega^2\right)^2 - 1}.$$

Both a_1 and a_2 go to infinity at $\omega = 1$ and $\omega = \sqrt{3}$, corresponding to normal-mode frequencies. At $\omega = \sqrt{2}$, $a_1 = 0$, corresponding to a motion where mass 2 vibrates with just the right amplitude so the total force on mass 1 is zero.

41. a) $$-kx_1 + k'(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) = m\ddot{x}_1,$$

$$-kx_2 - k'(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) = m\ddot{x}_2.$$

b) Substitute normal-coordinate expressions: $x_1 = q_1 + q_2$, $x_2 = q_1 - q_2$:

$$-kq_1 - (k + 2k')q_2 - 2b\dot{q}_2 = m(\ddot{q}_1 + \ddot{q}_2),$$
$$-kq_1 + (k + 2k')q_2 + 2b\dot{q}_2 = m(\ddot{q}_1 - \ddot{q}_2).$$

Add the two equations, and subtract the second from the first, to obtain

$$m\ddot{q}_1 + kq_1 = 0, \qquad m\ddot{q}_2 + 2b\dot{q}_2 + (k + 2k')q_2 = 0.$$

c) As before, in notes, Eq. (6), $\omega_1 = \sqrt{\dfrac{k}{m}}, \qquad \omega_2 = \sqrt{\dfrac{k + 2k'}{m}}.$

But now the general solutions of the q equations are

$$q_1 = C\cos(\omega_1 t + \varphi_1), \quad q_2 = e^{-\gamma_2 t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right], \quad \text{where}$$

$$\gamma_2 = \frac{2b}{2m} = \frac{b}{m} \quad \text{and} \quad \omega_d = \sqrt{\omega_2^{\,2} - \gamma_2^{\,2}}.$$

Interpretation: In the q_1 mode there is no relative motion of the masses, so this mode is not damped. In the q_2 mode, the distance between the two ends of the shock absorber changes by twice as much as the motion of each individual mass, hence the factor of 2 in the damping term of the q_2 equation.

So the general solution for each mass consists of an undamped oscillation in which the two masses vibrate in unison with equal amplitudes and frequency ω_1 and a damped oscillation of one mass relative to the other, characterized by the quantities ω_2 and γ_2. This motion is underdamped if $\omega_2 > \gamma_2$ (as in the above solution), critically damped if $\omega_2 = \gamma_2$, or overdamped, if $\omega_2 < \gamma_2$

For example, if the masses are given equal initial displacements and no initial velocities, then $q_2 = 0$ and only the undamped motion is present. But with opposite initial displacements of equal magnitude, $q_1 = 0$, and only the damped motion occurs.

42.

a) $$\mathbf{AB} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{pmatrix}, \qquad \mathbf{BA} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{pmatrix}.$$

b) $$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 1 & -1 \\ 4 & 0 & 2 \end{pmatrix}.$$

c) $$\mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & -1 & \frac{1}{4} \\ -\frac{1}{4} & -1 & \frac{3}{4} \end{pmatrix}; \qquad \mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) $$4\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -4 & 1 \\ -1 & -4 & 3 \end{pmatrix}.$$