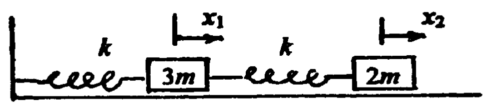
33-231 Exam IV December 4, 2002

1. (30 points = 12 + 18) Name________________________________

1	
2	
3	
4	
total	

a) For the system shown, obtain the **M** and **K** matrices, using the coordinates shown.



$\sum F = ma$ equations:

$$3m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$

$$2m\ddot{x}_2 = -k(x_2 - x_1) = kx_1 - kx_2$$

$$\underline{\underline{M}} = \begin{pmatrix} 3m & 0 \\ 0 & 2m \end{pmatrix}, \quad \underline{\underline{K}} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$$

(continued)

1. (continued)

b) A certain oscillating system, not the same one as in part (a), has the following

M and **K** matrices: $\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $\mathbf{K} = \begin{pmatrix} 5k & -2k \\ -2k & 2k \end{pmatrix}$.

Use these matrices to obtain the normal-mode frequencies and eigenvectors. Describe in words the amplitude relations for each mode.

$$|\underline{\underline{K}} - \lambda \underline{\underline{M}}| = 0: \quad \begin{vmatrix} 5k-\lambda m & -2k \\ -2k & 2k-\lambda m \end{vmatrix} = 0$$

$$(5k-\lambda m)(2k-\lambda m) - 4k^2 = 0$$

$$6k^2 - 7k\lambda m + \lambda^2 m^2 = 0, \quad (6k-\lambda m)(k-\lambda m) = 0$$

$$\lambda_1 = \frac{k}{m}: \quad \begin{pmatrix} 5k-\left(\frac{k}{m}\right)m & -2k \\ -2k & 2k-\left(\frac{k}{m}\right)m \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 4k & -2k \\ -2k & k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow 4ka_1 - 2ka_2 = 0, \quad 2a_1 = a_2$$

$$\underline{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(Masses move in phase; second mass has twice the amplitude of first)

$$\lambda_2 = \frac{6k}{m}: \quad \begin{pmatrix} 5k-\left(\frac{6k}{m}\right)m & -2k \\ -2k & 2k-\left(\frac{6k}{m}\right)m \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -k & -2k \\ -2k & -4k \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \Rightarrow b_1 = -2b_2, \quad \underline{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Masses move $\frac{1}{2}$ cycle out of phase; first mass has twice the amplitude of second. $\omega_1 = \sqrt{\frac{k}{m}}$, $\omega_2 = \sqrt{\frac{6k}{m}}$

2. (20 points = 7 + 7 + 6)

A flexible rope has a total length of 2.00 m and a total mass of 0.600 kg. It is stretched with a tension of 120.0 N.

a) Find the speed of transverse waves on the rope. Carry units through your calculation and verify that they are consistent.

$$F = 120.0\,N, \quad \mu = \frac{0.600\,kg}{2.00\,m} = 0.300\,kg/m$$

$$c = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{120.0\,N}{0.300\,kg/m}} = \sqrt{\frac{120.0\,kg\,m/s^2}{0.300\,kg/m}}$$

$$= \sqrt{400\,m^2/s^2} = \boxed{20.0\,m/s}$$

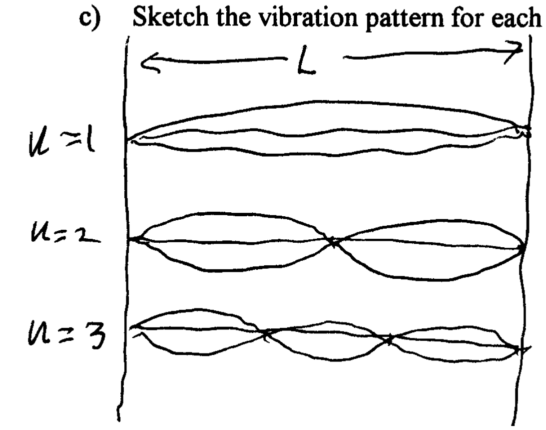
b) If the ends of the rope are held stationary, find the frequencies of the three normal modes that have the lowest frequencies.

$$L = n\frac{\lambda}{2}\ ; \quad \lambda = \frac{c}{f}\ ; \quad L = \frac{nc}{2f}, \quad f = \frac{nc}{2L}$$

$$n=1: \quad f_1 = \frac{(1)(20\,m/s)}{2(2.00\,m)} = 5.00\,s^{-1} = 5.00\,Hz$$

$$n=2: \quad f_2 = 10.0\,Hz \qquad n=3: \quad f_3 = 15.0\,Hz$$

c) Sketch the vibration pattern for each of the three modes in (b).



3. (30 points = 10 + 10 + 10)

Two ropes lie along the x axis; they are tied together at the point $x = 0$ and are stretched with tension F. The rope in the $x < 0$ region has linear mass density μ_1, and the rope in the $x > 0$ region has linear mass density μ_2. A sinusoidal incident wave with angular frequency ω, wave number k, and amplitude A travels in the positive x direction toward the point $x = 0$.

a) Write an expression for the total wave function in the region $x < 0$ and an expression for the total wave function in the region $x > 0$. Let the amplitudes of the incident, reflected, and transmitted waves be A, B, and C, respectively

$$x<0: \; y_-(x,t) = A\cos(k_1 x - \omega t) + B\cos(k_1 x + \omega t)$$

$$x>0: \; y_+(x,t) = C\cos(k_2 x - \omega t)$$

b) State the boundary conditions that must be satisfied at the point $x = 0$; explain briefly the basis of each.

① at $x=0$, $y_-(0,t) = y_+(0,t)$ because ends of the two pieces have to meet at $x=0$

② at $x=0$, $\left(\frac{\partial y_-}{\partial x}\right)_{x=0} = \left(\frac{\partial y_+}{\partial x}\right)_{x=0}$.

At junction, the two ropes must have the same slope because the knot is massless and total transverse F must be zero.

(continued)

3. (continued)

c) Use the boundary conditions in part (b) to obtain equations that can be solved to obtain expressions for the amplitudes B and C as multiples of A. Do not solve these equations, but write them in forms that contain only μ_1, μ_2, A, B, and C.

(1) $A\cos(-\omega t) + B\cos(\omega t) = C\cos(-\omega t)$ $\quad \cos(-\alpha) = \cos\alpha$, $\sin(-\alpha) = -\sin\alpha$

$$\boxed{A + B = C}$$

(2) $-Ak_1\sin(-\omega t) - Bk_1\sin(\omega t) = Ck_2\sin(-\omega t)$

$$Ak_1 - Bk_1 = Ck_2$$

(ω must be the same for both sides)

$$k_1 = \frac{\omega}{v_1} = \omega\sqrt{\frac{\mu_1}{F}}$$

$$k_2 = \frac{\omega}{v_2} = \omega\sqrt{\frac{\mu_2}{F}}$$

so $A\omega\sqrt{\frac{\mu_1}{F}} - B\omega\sqrt{\frac{\mu_1}{F}} = C\omega\sqrt{\frac{\mu_2}{F}}$

$$\boxed{A\sqrt{\mu_1} - B\sqrt{\mu_1} = C\sqrt{\mu_2}}$$

4. (20 points = 10 + 5 + 5)

a) In Problem 3, assume that A, B, and C are known. Derive an expression for the instantaneous rate of energy flow past the point $x = 0$.

$$P = -F \frac{\partial y_T}{\partial x}\frac{\partial y_T}{\partial t}; \quad \text{at } x = 0,$$

$$= -F(-Ck_1 \sin(-\omega t))(+C\omega \sin(-\omega t))$$

$$= FC^2 k_1 \omega \sin^2 \omega t = \boxed{FC^2 \frac{\omega^2}{c} \sin^2 \omega t}$$

b) Show that the instantaneous energy flow is always in the direction of increasing x.

P expression must be non-negative because $\sin^2 \omega t$ is non-negative and the parameters are always positive

c) If the angular frequency of the incident wave is doubled, with μ_1, μ_2, F, and A remaining the same, by what factor does the rate of energy flow change? Explain your answer briefly.

C doesn't change; P contains ω^2, so when ω doubles, P increases by a factor of 4.