33-232 Exam III November 10, 2000

1. (30 points = 8 + 7 + 8 + 7) Name______________________

1	
2	
3	
4	
total	

A particle with mass m moves along the x axis under the action of a force described by the function $F = 12x(2-x-x^2)$.

a) If the potential energy is zero at $x = 0$, obtain an expression for the potential energy as a function of x.

$F = -\frac{dV}{dx}$, $V = -\int F\,dx$

$= -12x^2 + 4x^3 + 3x^4 + C$

But $V(0) = 0$, so $C = 0$ and

$$\boxed{V = -12x^2 + 4x^3 + 3x^4}$$

$\frac{d^2V}{dx^2} = -24 + 24x + 36x^2$

(needed for part (b).)

b) Find all the possible equilibrium positions for the particle, and determine which ones are stable and which are unstable.

Equilibrium: $F = 0$ $F = 12x(2+x)(1-x)$

$x = -2, 0, 1$ (or solve $2 - x - x^2 = 0$ using quadratic formula)

$\frac{d^2V}{dx^2} > 0 \Leftrightarrow$ stable equilibrium; $\frac{d^2V}{dx^2} < 0$, unstable

$$\boxed{\begin{aligned} x = -2,\ \frac{d^2V}{dx^2} &= 72 \text{ (stable)} \\ x = 0,\ \frac{d^2V}{dx^2} &= -24 \text{ (unstable)} \\ x = 1,\ \frac{d^2V}{dx^2} &= +36 \text{ (stable)} \end{aligned}}$$

(continued)

1. (continued)

c) One of the stable equilibrium positions occurs at a positive value of x. If the particle is placed near this position and given a small initial velocity, it oscillates about this position. Find the angular frequency of small oscillations about this position.

Effective force constant:

$$"k" = -\left[\frac{dF}{dx}\right]_{x=x_0} = \left[\frac{d^2V}{dx^2}\right]_{x=x_0} \qquad (x_0 = 1)$$

$$\left[\frac{d^2V}{dx^2}\right]_{x=1} = 36$$

$$\boxed{\omega = \sqrt{\frac{"k"}{m}} = \sqrt{\frac{36}{m}}}$$

d) If the particle is placed at the point $x = 0$ and given a very small negative initial velocity, derive an expression for its velocity when it reaches the point $x = -1$.

$$K_0 + V_0 = K_{-1} + V_{-1}$$

$$V_{-1} = -12(-1)^2 + 4(-1)^3 + 3(-1)^4$$
$$= -13$$

$$0 + 0 = \frac{1}{2}mv^2 + (-13)$$

$$\boxed{v = \pm\sqrt{\frac{26}{m}}}$$

2) (30 points = 15 + 5 + 10)

A vibrating tuning fork is electrically driven by an electromagnet that applies a sinusoidal driving force. The system can be modeled as a lightly damped, driven harmonic oscillator. The angular frequency of free oscillation is $\omega_o = 1500\ s^{-1}$, and $Q = 250$. When the angular frequency of the driving force is $1500\ s^{-1}$, the amplitude of the forced oscillation is 2.00 mm. Then the angular frequency of the driving force is increased by an amount $\Delta\omega$. The amplitude of the forced oscillations is then found to be 1.20 mm.

a) Determine the frequency increase $\Delta\omega$. Small Damping: $A' \simeq \dfrac{F_o/2m\omega_o}{\sqrt{(\omega_o-\omega)^2+\gamma^2}}$

$$Q = \frac{\omega_o}{2\gamma} \Rightarrow \gamma = \frac{\omega_o}{2Q} = \frac{1500\,s^{-1}}{2(250)} = 3\,s^{-1}$$

$\Delta\omega = \omega - \omega_o$

$$\frac{A'_{1500+\Delta\omega}}{A'_{1500}} = \frac{\sqrt{\gamma^2}}{\sqrt{(\Delta\omega)^2+\gamma^2}} = \frac{1.20\,mm}{2.00\,mm} = \frac{6}{10}$$

$$10\gamma = 6\sqrt{\Delta\omega^2+\gamma^2}\ ,\quad 100\gamma^2 = 36(\Delta\omega^2+\gamma^2)$$

$$\Delta\omega^2 = \frac{64}{36}\gamma^2 \Rightarrow \Delta\omega = \frac{8}{6}\gamma = \frac{8}{6}(3\,s^{-1})$$

$$\boxed{\Delta\omega = 4\,s^{-1}}$$

b) When the driving frequency has its original value of $1500\ s^{-1}$, what is the *phase* of the forced oscillation relative to that of the driving force?

$$\tan\phi = \frac{2\gamma\omega}{\omega^2-\omega_o^2}.$$

as $\omega \to \omega_o$, $\tan\phi \to -\infty$,

$$\boxed{\phi \to -\frac{\pi}{2}}$$

(continued)

2. (continued)

c) Suppose we increase Q to 1000. Describe how the amplitudes of the forced oscillations for the two cases ($\omega = 1500\ s^{-1}$ and $1500\ s^{-1} + \Delta\omega$) change. Your answers should be quantitative, but do not make detailed calculations.

amplitude at $\omega = 1500\ s^{-1}$

increases by factor of 4

because γ creases by a factor of 4.

amplitude at $\omega = 1503\ s^{-1}$ is

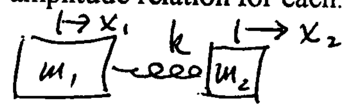
approximately the same because

now γ^2 is much smaller than $\Delta\omega^2$

3. (25 points = 15 + 10)

Two railroad cars, with masses m_1 and m_2, roll without friction along a straight level track. They are connected by a spring with force constant k.

a) Determine the frequencies of all the normal modes of the system, and obtain the amplitude relation for each.



$$m_1\ddot{x}_1 = k(x_2 - x_1) = kx_2 - kx_1$$
$$m_2\ddot{x}_2 = -k(x_2 - x_1) = kx_1 - kx_2$$

$x_1 = A_1 \cos\omega t,\quad x_2 = A_2 \cos\omega t$

$$-m_1\omega^2 A_1 = kA_2 - kA_1 \qquad (k - m_1\omega^2)A_1 - kA_2 = 0$$
$$-m_2\omega^2 A_2 = kA_1 - kA_2 \qquad -kA_1 + (k - m_2\omega^2)A_2 = 0$$

Non-trivial solutions is $\begin{vmatrix} k - m_1\omega^2 & -k \\ -k & k - m_2\omega^2 \end{vmatrix} = 0$

$$(k - m_1\omega^2)(k - m_2\omega^2) - k^2 = 0$$
$$(m_1 + m_2)k\omega^2 = (m_1 m_2)\omega^2 \qquad \omega^2 = 0,\ \frac{k(m_1+m_2)}{m_1 m_2}$$

$\omega_1 = 0,\quad \omega_2 = \sqrt{\dfrac{k(m_1+m_2)}{m_1 m_2}}$

$\omega_1 = 0,\quad A_1 = A_2$

$\omega_2 = \sqrt{\dfrac{k(m_1+m_2)}{m_1 m_2}},\quad \left(k - \dfrac{m_1(m_1+m_2)}{m_1 m_2}k\right)A_1 = kA_2 \rightarrow A_2 = -\dfrac{m_1}{m_2}A_1$

b) Obtain a normal-coordinate transformation for the system.

$$\omega_1 \downarrow \qquad \downarrow \omega_2$$

$$x_1 = \xi_1 + \xi_2$$

$$x_2 = \xi_1 - \frac{m_1}{m_2}\xi_2$$

4. (15 points = 8 + 7)

This problem is about the logistic map: $x_{n+1} = ax_n(1 - x_n)$.

a) For $a = 2$, find all the values of x that could be single-cycle attractors or repellers.

For repeller or attractor, $x_{n+1} = x_n$,

If $a = 2$, $x = 2x(1-x)$;

$2x^2 - x = 0$, $x(2x-1) = 0$

$\boxed{x = 0,\ x = \frac{1}{2}}$

b) In our class discussion, the values of a were limited to a particular range. Determine this range, and show why this limitation is necessary.

We restricted x to the range $0 \leq x \leq 1$. The maximum value of $x(1-x)$ in this range is $\frac{1}{4}$. If $a > 4$, this range will be exceeded.

Useful Equations:

$$T = 2\pi\sqrt{\frac{m}{k}}, \qquad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \qquad \omega_o = \sqrt{\frac{k}{m}}, \qquad \gamma = \frac{b}{2m} \tag{6), (7}$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_o{}^2 x = 0. \tag{11}$$

$$x = Ae^{-(\gamma+\gamma_d)t} + Be^{-(\gamma-\gamma_d)t} = e^{-\gamma t}\left(Ae^{-\gamma_d t} + Be^{\gamma_d t}\right), \tag{12}$$

$$A = -\frac{(\gamma - \gamma_d)x_o + v_o}{2\gamma_d}, \qquad B = \frac{(\gamma + \gamma_d)x_o + v_o}{2\gamma_d}. \tag{13}$$

$$x = e^{-\gamma t}\left(A\cos\omega_d t + B\sin\omega_d t\right), \tag{14}$$

$$A = x_o, \qquad B = \frac{v_o + \gamma x_o}{\sqrt{\omega_o{}^2 - \gamma^2}} = \frac{v_o + \gamma x_o}{\omega_d}. \tag{15}$$

$$x = (A + Bt)e^{-\gamma t}. \tag{16}$$

$$A = x_o, \qquad B = \gamma x_o + v_o. \tag{17}$$

$$\left|\frac{\Delta E}{E}\right| \cong \frac{4\pi\gamma}{\omega_d} \cong \frac{4\pi\gamma}{\omega_o}. \tag{27}$$

$$\left|\frac{\Delta E \text{ (per radian)}}{E}\right| = \frac{2\gamma}{\omega_o}. \tag{28}$$

$$Q = \frac{\omega_o}{2\gamma}. \tag{29}$$

$$E = E_o e^{-2\gamma t}. \tag{30}$$

$$A' = \frac{F_o/m}{\omega_o{}^2 - \omega^2}; \qquad \text{then} \qquad x = \frac{F_o/m}{\omega_o{}^2 - \omega^2}\cos\omega t. \tag{33}$$

$$A' = \frac{F_o/m}{\sqrt{\left(\omega^2 - \omega_o{}^2\right)^2 + (2\gamma\omega)^2}}, \qquad \tan\varphi = \frac{2\gamma\omega}{\omega^2 - \omega_o{}^2}. \tag{37}$$

$$A' \cong \frac{F_o/m}{\sqrt{(\omega - \omega_o)^2(\omega_o + \omega_o)^2 + (2\gamma\omega_o)^2}} = \frac{F_o/2m\omega_o}{\sqrt{(\omega_o - \omega)^2 + \gamma^2}}. \tag{41}$$

$$x = e^{-\gamma t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + A'\cos(\omega t + \varphi). \tag{45}$$