

**Problems:** Set 9 (due Wednesday, October 29, 2003)

35. This problem continues the analysis of the two-well potential-energy function of Problems 33 and 34. Now we add a periodic driving force. We'll use numerical solutions of the differential equation to experiment with some of the parameters of the problem and observe the resulting motion. Use the same numerical values as in Problems 33 and 34, namely,  $a = 1$ ,  $m = 1$ ,  $k = 1$ , and  $b = 0.1$ .

Note: We will be looking for limit cycles in the phase trajectories. Sometimes the driving force has to go through many cycles before the limit cycle becomes apparent. In some cases you may have to take  $t$  up to 500 or so, and plot 1000 or more points, to see the limit cycle. It may take a little time for Maple to plot the phase trajectory, depending on your processor speed. Be patient! Don't print out *all* your plots, only one or two representative ones for each part of the problem.

- a) For the situation of Problem 34, we add a sinusoidal driving force  $F = F_o \cos \omega t$ . Write the general differential equation (from  $\Sigma F = ma$ ).
- b) Obtain a numerical solution and a phase plot for the equation obtained in (a). As an initial trial, I suggest  $F_o = 0.02$  and  $\omega = 0.2$ . Take the initial conditions to be  $x_o = 0$ ,  $v_o = 0$ . You should find a limit cycle, but note the wonderfully circuitous route the phase point takes in approaching the limit cycle. Also make a plot of  $x$  vs.  $t$ , which shows a different representation of the approach to the limit cycle. Be sure to plot enough points to obtain smooth curves.
- c) In view of the result of (b), can you suggest a set of initial conditions that would make the system approach the limit cycle more quickly? Try them and see whether they have the expected result.
- d) Set the initial conditions back to  $x_o = 0$ ,  $v_o = 0$ , and increase  $F_o$ . I suggest increasing it in steps of 0.02, up to 0.10. You may want to indulge in the luxury of a few runs with  $t$  up to 1000 and `numpoints = 5000` or `10000`. Do you reach a point where there no longer seems to be a limit cycle? How can you be sure a limit cycle wouldn't appear if you ran the solution long enough?
- e) Now try increasing  $F_o$  still more and experiment with small variations in initial conditions, to explore the question whether the motion is very sensitive to small changes in initial conditions. Summarize your results in words. For the grand finale, try  $F_o = 0.4$ ,  $\omega = 0.4$  (the values used in class on October 22).

36. The logistic map, which we discussed in class, has played a very important role in the study of population growth, nonlinear dynamics, and many other areas. The map is expressed by the equation  $x_{n+1} = ax_n(1 - x_n)$ .
- If the first element in the sequence (i.e.,  $x_0$ ) is in the range  $0 \leq x_0 \leq 1$ , show that all succeeding  $x_n$ 's in the sequence also lie in this range, provided that  $0 \leq a \leq 4$ .
  - For any given value of  $a$ , derive an expression for the value (or values) of  $x_n$  such that  $x_{n+1} = x_n$ .
  - For given values of  $a$  and  $x_0$ , we want to compute a list of  $x_n$ 's, large enough to determine whether they seem to be approaching an attractor. There are several ways to do this using Maple. The simplest is to compute a sequence of values. If we choose  $a = 2.5$  and compute 20 values, starting with  $x = 0.5$ ,

```
restart;
a := 2.5;  x := 0.5;  N := 20;
for n from 1 to N do
x := a*x*(1 - x);
end do;
```

The variable  $N$  counts the number of iterations; each value of  $x$  is computed by substituting the previous value of  $x$  into the logistic equation.

A somewhat more elegant way is to use the notation  $x[n]$ , in which Maple interprets the quantity in square brackets as a subscript, as in  $x_n$ . Thus:

```
restart;
a := 2.5;  x[0] := 0.5;  N := 20;
for n from 0 to N do
x[n + 1] := evalf(a*x[n]*(1 - x[n]));
end do;
```

Still nicer is to plot a graph showing  $x_n$  as a function of  $n$ . The Maple plot command will plot a list of points, if you give it a list of pairs of numbers (coordinates) for the points. (Note that each pair is itself a list.) For this problem we can use `seq` to generate a list of coordinate pairs,  $[n, x[n]]$ , which we'll call `pointlist`. That is, `pointlist := seq([n, x[n]], n = 0..N)`: All together, it looks like this:

```
restart;
a := 2.5,  x[0] := 0.5;  N := 20;
for n from 0 to N do
x[n + 1] := evalf(a*x[n]*(1 - x[n]));
end do:
pointlist := seq([n, x[n]], n = 0..N):
plot([pointlist]);
```

(continued)

36. (continued)

I suggest placing semicolons at the ends of lines 4 through 6 until you are sure it's working; then replace them with colons to suppress the information you don't need. That's important when you need to let  $n$  run up to several hundred.

Maple will draw lines between the points unless you tell it not to. Try using the plot options `style = point` and `symbol = cross`. The "plot" command is then `plot([pointlist], style = point, symbol = cross)`; Read the Maple help file `?plot[options]` for details.

Also note that "pointlist" has to be in square brackets to show that it is indeed a list. Alternatively, we could put the right side of the next to last line in square brackets:

$$\text{pointlist} := [\text{seq}([n, x[n]], n = 0..N)]:$$

Then the square brackets aren't needed in the "plot" command.

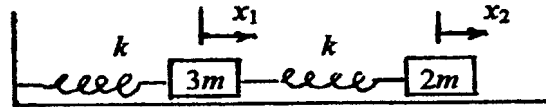
Now back to the problem. Choose a value of  $a$  between 1 and 2.5, and try several different values of  $x_0$  to check whether the attractor is independent of  $x_0$ . Then try a different value of  $a$  and again find the attractor. Compare your results with predictions obtained from the result of (b).

- d) Take  $a = 3.1$ , and obtain an equation for the value of  $x_0$  such that for each  $x$ ,  $x_{n+2} = x_n$ . This will be a fourth-degree equation; use Maple to find the roots of this equation that lie in the range  $0 \leq x \leq 1$ . You should find two attractors.
- e) For  $a = 3.1$ , choose a few values of  $x_0$  and repeat the calculations of (c) for each. Compare the attractors you find with your predictions from (d).
- f) By trial and error, find the value of  $a$  for which the *second* bifurcation occurs and the sequence of  $x$ 's converges to a repeating cycle of *four* attractors.
- g) Show that when  $a = 3$ , the line  $y = x$  is perpendicular to the tangent to the curve  $y = ax(1 - x)$  at the point of intersection. What does this tell you about whether the value of  $x$  at this point is an attractor or a repeller?

37. Consider the mapping  $x_{n+1} = a \sin(\pi x_n)$ .

- Find the permissible values of  $a$  such that every  $x_n$  is in the range  $0 \leq x_n \leq 1$  if  $x_0$  is in that range.
- If  $a = 0.6$ , use the Maple `fsolve` command to find the values of  $x$  such that  $x_{n+1} = x_n$ . Are these values attractors or repellers?
- Show that if  $a = 0.5$ , then  $x = 0.5$  is an attractor.
- By trial and error, show that bifurcations occur at sufficiently large values of  $a$ . The first is around 0.72, and the next three are between 0.83 and 0.87. Find the values of  $a$  for the first three bifurcations, to three significant figures.

38. a) For the system shown, write the  $\Sigma F = ma$  equations, using the coordinates shown.



- From these equations, find the frequencies of the normal modes and the amplitude relation for each normal mode.
- Obtain a normal-coordinate transformation. That is, obtain equations for  $x_1$  and  $x_2$  in terms of  $q_1$  and  $q_2$ . Also obtain equations for the *inverse* transformation (the equations for  $q_1$  and  $q_2$  in terms of  $x_1$  and  $x_2$ ).
- Express the  $\Sigma F = ma$  equations in terms of the normal coordinates, and show that they can be separated into two uncoupled equations, each of which contains only one normal coordinate. Solve the equations for the normal coordinates, and verify that you get the same normal-mode frequencies as in part (b).
- Write an expression for the total energy of the system in terms of  $x_1$ ,  $x_2$ , and their derivatives. Then express this quantity in terms of the normal coordinates, and show that it can be separated into two parts, each of which contains only one normal coordinate.
- Express the normal-coordinate transformation in terms of a matrix equation.

That is, let  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ , and find a matrix  $\mathbf{A}$  such that

$\mathbf{x} = \mathbf{A}\mathbf{q}$ . Find  $\mathbf{A}^{-1}$  (the inverse of  $\mathbf{A}$ ), and verify that the transformation

$\mathbf{q} = \mathbf{A}^{-1}\mathbf{x}$  gives the same equations as those obtained in (c).