<u>Problems:</u> Set 8 (due Wednesday, October 22, 2003)

33. Problems 33 through 35 involve several aspects of the behavior of a particular system, including phase plots, vector field plots, attractors, limit cycles, sensitivity to initial conditions, and chaotic (non-periodic) behavior with periodic driving forces.

    The system consists of a particle with mass m moving along a straight line (coordinate x) under the action of a force described by a potential-energy function

    $$V(x) = \frac{1}{2}kx^2\left(1 - \frac{x}{a}\right)^2, \qquad \text{where } k \text{ and } a \text{ are positive constants.}$$

    In the following discussion, answer the questions in terms of the symbolic quantities m, k, and a whenever possible. But when you have to plot a function or obtain a numerical solution for a differential equation, you need to substitute specific *numerical* values. In such cases, make the simplest possible choices, namely $m = 1$, $k = 1$, $a = 1$.

    a) Using the suggested numerical values, plot a graph of $V(x)$. To see the shape of the curve clearly, I suggest the range $x = -0.3..1.3$.

    b) For general (symbolic) values, obtain an expression for the force $F(x)$ acting on the particle. Show that there are three equilibrium points, two stable and one unstable. Find the value of x for each, and find the value of $V(x)$ at each equilibrium point.

    c) For each *stable* equilibrium point, obtain a Taylor series expansion of $F(x)$ about the point. Keeping only the first term in the expansion, determine an effective force constant and obtain symbolic expressions for the angular frequency and period of small oscillations about the equilibrium position. Then obtain numerical values, using the numbers suggested above.

    d) Write the appropriate differential equation for the motion (from $\Sigma F = ma$). Using the exact expression for $F(x)$ from (b) (i.e., not the Taylor series approximation), with the numerical values suggested above, obtain a numerical solution that represents a small oscillation near the stable equilibrium position $x = 0$. I suggest taking initial conditions $x_o = 0$ and a small v_o, e.g., $v_o = 0.01$.

       Plot a graph of x as a function of t, and also a phase plot of v as a function of x. From the x vs. t graph, estimate the period of the motion, and compare with the predictions you obtained in (c). Does the phase trajectory have the expected shape?

(continued)

33. (continued)

e) In Part (d), try larger values of v_o, up to 0.2 or so; see how the period and the shape of the phase plot depend on v_o.

f) Suppose we take $x_o = 0$. Using results from (b), derive a general symbolic expression for the minimum value of v_o needed for the particle to pass the unstable equilibrium point and approach the other stable equilibrium point.

Using numerical values, obtain a numerical solution of the differential equation for a value of v_o slightly larger than this minimum. Obtain a graph of x vs. t and a phase plot for this situation. Experiment with various values of v_o. Can you make the particle *stop* at the unstable equilibrium point? If v_o is close to the critical value, is the resulting motion very sensitive to small changes in v_o?

g) Obtain a field plot for the system, and sketch out some of the possible phase trajectories you have obtained in preceding parts. You may have to experiment with various ranges of x and v to get useful plots. Compare your sketches to the phase plots you obtained above.

34. Now we add a damping force $F = -bv$ to the system of Problem 33.

a) Obtain the differential equation for the motion. Using the numerical values suggested in Problem 33, and assuming $b = 0.1$, obtain a numerical solution of the differential equation, assuming $x_o = 0$ and $v_o = 0.1$. Obtain a graph of x vs. t and a phase plot. Each graph should show several oscillations.

b) Repeat (a), using the initial conditions $x_o = 1$, $v_o = 0.1$. Compare your results with the results from (b).

c) Considering the analysis of Problem 33(f), do you expect a significant change in behavior of the system when v_o is greater than the critical value found there? Test your prediction by obtaining a graph of x vs. t and a phase plot for $v_o = 0.3$.

d) Repeat the analysis of (c), using $v_o = 1.0$ and using $v_o = 1.1$. You should find that the particle is attracted to one stable equilibrium point when $v_o = 1.0$ and to the other when $v_o = 1.1$. By trial and error, find (to at least three significant figures) the critical value of v_o for which the "cross-over" occurs. Note the extreme sensitivity of the motion to very small changes in v_o near this critical value.

35. (in next week's assignment)