Problems: Set 7 (due Wednesday, October 15, 2003)

30. a) The *average value* of a function $f(x)$ over an interval $x = a \,..\, b$ is defined as

$$\big[f(x)\big]_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This is often abbreviated as $\big[f(x)\big]_{\text{ave}} = \langle f \rangle$ or $\bar{f}$.

Show that the average value of the quantity $\sin^2 \omega t$ or $\cos^2 \omega t$, averaged over a complete cycle or any integer number of complete cycles, is 1/2. Also show that the average value of $\sin \omega t \cos \omega t$, over a complete cycle, is zero. That is,

$$\langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle = \frac{1}{2}, \qquad \langle \sin \omega t \cos \omega t \rangle = 0,$$

where, as above, the $\langle \quad \rangle$ angle brackets denote "average value."

b) Review the definition of *power* in any introductory text. In particular, when a force F acts on a body that moves on the same line as the force with speed v, the power P (instantaneous time rate of doing work) is given by $P = Fv$. (In general both F and v vary with time.) Hence show that for a damped, sinusoidally driven harmonic oscillator, the instantaneous input power supplied by the driving force is $P_{\text{in}} = Fv$, and the instantaneous rate of dissipation of energy by the damping force is $P_{\text{out}} = -bv^2$. (This is fairly trivial.)

c) In steady-state oscillations, the *average* input power from the driving force must equal the *average* power dissipated by the damping force. Show that for a lightly damped system the average power dissipated, $\langle P_{\text{out}} \rangle$, is

given approximately by $\displaystyle \langle P_{\text{out}} \rangle = \frac{F_o^{\,2}\gamma}{4m}\,\frac{1}{(\omega-\omega_o)^2+\gamma^2}$. From this, show

that if the angular frequency ω of the driving force is varied (with F_o constant), the average power dissipated reaches a maximum at a certain frequency. Find this frequency, and show that at this frequency the average power dissipated is

$$\langle P_{\text{out}} \rangle = \frac{F_o^{\,2}}{4m\gamma}.$$

d) Discuss why the average power should be expected to be proportional to $F_o^{\,2}$, and why the frequency at which the maximum occurs might have been expected.

e) (optional) The average *input* power $\langle P_{\text{in}} \rangle$ can also be computed. It's a little more complicated because F and v aren't in phase and you need to find the sine and cosine of the phase angle φ. But you can then show directly that the average input and output powers are equal.

31. For a damped, sinusoidally driven harmonic oscillator, the amplitude A' of forced oscillations is given by Eq. (37) in Chapter 5 of the notes. If the angular frequency ω of the driving force is varied, the amplitude $A'(\omega)$ reaches a maximum when

$\omega = \sqrt{\omega_o^2 - 2\gamma^2}$. (This is Eq. (39) in the notes.)

a) Show that the instantaneous *velocity* is given by $v(t) = -v_{max}\sin(\omega t + \varphi)$, with $v_{max} = \dfrac{\omega F_o/m}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2\gamma\omega)^2}}$; v_{max} may be called the *velocity amplitude*; it is the maximum magnitude of velocity during a cycle.

b) The velocity amplitude v_{max} depends on the frequency ω of the driving force. Find the frequency at which the *greatest* value of v_{max} occurs. Note: You first have to take the derivative of $x(t)$ with respect to t to find $v(t)$ and thus v_{max}, then take the derivative of v_{max} with respect to ω, which is messy. I suggest that you use Maple to do the derivative with respect to ω.

c) Find the value of v_{max} when ω has the value found in part (b).

32. The Van der Pol oscillator is an idealized model that has been used to study self-excited oscillations in mechanical systems (such as the old Tacoma Narrows Bridge) and also in some non-linear electric circuits (e.g., a resonant L-C circuit containing a tunnel diode). In its simplest form, the model consists of a particle moving in a straight line, with coordinate x, according to the equation

$$\frac{d^2x}{dt^2} = -x + (1 - x^2)\frac{dx}{dt}.$$

a) Show that the damping force has the expected direction when x is large, but that for small x it has the *same* direction as v and is thus an "anti-damping" force. What consequences do you think this will have for the motion?

b) Study the motion of this system, using x vs t plots and phase plots with numerical solutions of the differential equation. Try a variety of initial conditions to look for a limit cycle. Determine the approximate period of the limit-cycle motion.

c) Re-write the differential equation as a pair of coupled first-order equations for the variables x and v. Using these equations, obtain a vector field plot, and show that it suggests the existence of a limit-cycle attractor that is independent of the initial conditions.

d) Using the equations in (c), show that the point ($x = 0$, $v = 0$) must be a repeller, not an attractor.